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Discrete Curvature (Draft), 2015-09-06

7.1 Length of Polygons

Definition 1 *The* length *of a polygonal curve* γ *with vertices* $\{p_0, ..., p_m\}$ *is the total length of all edges*

$$L(\gamma) := \sum_{i=1}^{m} |p_i - p_{i-1}| .$$

Definition 2 *Let* $c : I = [a, b] \to \mathbb{R}^n$ *be a parametrized curve. Let* S *be the set of all finite monoton subdivisions* $\sigma = (t_0, t_1, ...t_m)$ *of the interval* $I = [a, b]$ *such that*

$$a = t_0 < t_1 < ... < t_m = b.$$

Let $L(\sigma)$ *denote the length of the interpolating polygon* $c(t_0), ..., c(t_m)$. *If*

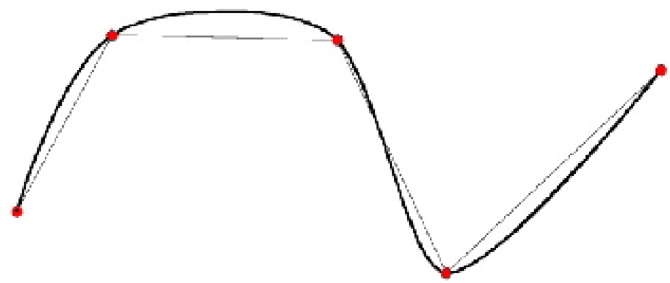
$$\sup_{\sigma \in S} L(\sigma)$$

is finite, then the curve c *is called* rectifiable *and its* length *is this supremum*

$$L(c) := \sup_{\sigma \in S} L(\sigma).$$

Remark 3 *Koch snow flake is not rectifiable.*

Theorem 4 *The length of a smooth curve $c : I = [a, b] \to \mathbb{R}^n$ can be determined from any sequence of monoton interpolatory polygons whose edge lengths uniformly tend to* 0*.*



Remark 5 *A non-interpolatory approximation of a line with a ZIG _ZAG polygon may certainly fail.*

7.2 Area of Surfaces

Definition 6 *Let M be a simplicial surface, then we define*

$$\operatorname{area} M := \sum_{\substack{T \subset M \\ T \ triangle}} \operatorname{area} T.$$

Theorem 7 *The area of a smooth surface can be determined from a sequence of interpolatory triangulations if a convergence of face normals is ensured.*

The assumed convergence of surface normals is essential as the counterexample of the Schwarz Lantern, named after Hermann Amandus Schwarz, demonstrates.

7.2.1 Schwarz Lantern

The Schwarz Lantern is a counterexample showing a situation where the approximation of a smooth surface M with a sequence of interpolating polyhedral surfaces $\{M_{h,i}\}$ may NOT imply convergence of area under certain subdivision conditions.

FIGURE 7.1. A round and a discretized cylinder.

A cylinder with radius r and height h has area:

$$\text{area}(\text{Cylinder}) = 2\pi r h. \tag{7.1}$$

Now we discretize the cylinder in the following way:

Definition 8 *A* Schwarz Lantern *interpolates a cylinder of height h and radius r with vertices of an (m, n) rectangular grid with m rectangles in vertical direction and n rectangles in angular direction. Every second horizontal ring is twisted by $\frac{\pi}{n}$, i.e. half of the angular discretization angle, and the rectangles then subdivided along their smaller diagonal into two triangles. This generates an alternating grid shown in figure 7.1.*

Lemma 9 *The limit of the area of a Schwarz Lantern is given by:*

$$\lim_{m,n\to\infty} \text{area}\, Lantern(m,n) = 2\pi r \cdot \sqrt{h^2 + \frac{r^2\pi^4}{4}\left(\lim_{m,n\to\infty}\frac{m}{n^2}\right)^2}$$

Note, the limit of the area depends on the quotient $\frac{m}{n^2}$!

Proof. a.) Area of triangle T: First let us compute the area for a triangle T. Two vertices A and B of the triangle lie on a circle of radius r enclosing a circular sector of angle $\alpha = \frac{2\pi}{n}$ centered at the cylinder axis. The length of the edge is $|AB| = 2r\sin\frac{\pi}{n}$.
The height h_C is the distance of the edge midpoint $e = \left(r\cos\frac{\pi}{n}, 0\right)$ to the vertex $C = \left(r, \frac{h}{m}\right)$ both given in (x, z)-coordinates. We calculate $h_C = \sqrt{\frac{h^2}{m^2} + r^2\left(1 - \cos\frac{\pi}{n}\right)^2}$ and obtain the area of T:

$$\text{area}\, T = r\sin\frac{\pi}{n}\sqrt{\frac{h^2}{m^2} + r^2\left(1 - \cos\frac{\pi}{n}\right)^2} \tag{7.2}$$

b.) Area of Lantern: Now we calculate the limit of the area of the Lantern with $2mn$ triangles using equations 7.4 and 7.5 as

$$\begin{aligned}\text{area}\,Lantern(m,n) &= 2mn\cdot r\sin\frac{\pi}{n}\cdot\sqrt{\frac{h^2}{m^2}+r^2\left(1-\cos\frac{\pi}{n}\right)^2}\\ &= 2r\cdot n\sin\frac{\pi}{n}\sqrt{h^2+m^2r^2\left(4\sin^4\frac{\pi}{2n}\right)}\\ &= 2r\cdot n\sin\frac{\pi}{n}\sqrt{h^2+\frac{m^2}{4n^4}r^2\left(2n\sin\frac{\pi}{2n}\right)^4} \qquad (7.3)\end{aligned}$$

c.) Limit area of Lantern: We derive two limit expressions: First, as $n\to\infty$ the polygon of n such edges approximates a full circle $2\pi r$, therefore we have

$$\lim_{n\to\infty} n\sin\frac{\pi}{n}=\pi. \qquad (7.4)$$

Second, using $\cos\alpha=\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}$ we obtain

$$\begin{aligned}1-\cos\frac{\pi}{n} &= 1-\cos^2\frac{\pi}{2n}+\sin^2\frac{\pi}{2n} \qquad (7.5)\\ &= 2\sin^2\frac{\pi}{2n}\end{aligned}$$

Using equations 7.4 and 7.5 we obtain the limit of the area of the Lantern with $2mn$ triangles:

$$\lim_{m,n\to\infty}\text{area}\,Lantern(m,n)=2\pi r\cdot\sqrt{h^2+\frac{r^2\pi^4}{4}\left(\lim_{m,n\to\infty}\frac{m}{n^2}\right)^2} \qquad (7.6)$$

□

The second summand of the root determines the limit behaviour of the area of the Lantern. The area will approach the area of the smooth cylinder if and only if $\lim_{n\to\infty}\frac{m}{n^2}=0$. For any other limit of $\frac{m}{n^2}$ the limit area of the Lantern is a higher value up to infinity. Choosing an appropriate limit of $\frac{m}{n^2}$ we may produce a sequence of Lanterns whose area approaches any value between the area of the cylinder and infinity.

We summarize: although for $m,n\to\infty$ the interpolating Lantern converges in the Hausdorff sense to a smooth cylinder, the area may not converge. This non-convergence result for the area of surfaces is in contrast to the convergence of the length of interpolating polygonal curves converging in Hausdorff sense to smooth limit curves.

7.3 Hausdorff Distance of Sets

Definition 10 *Let $m \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ a (non-empty) subset of $\mathbb{R}^n$. The* distance *of m from the set A is defined as*

$$d(m, A) := \inf_{a \in A} d(m, a).$$

For a fixed set A we get a distance function

$$\begin{aligned} d_A &: \mathbb{R}^n \to R_+ \\ d_A(m) &: = d(m, A). \end{aligned}$$

Definition 11 *Let A be any subset of $\mathbb{R}^n$ and $\varepsilon > 0$. The $\varepsilon-$*tube*, $\varepsilon-$*ball *or $\varepsilon-$*neighbourhood *of A is the set*

$$A_\varepsilon = \{m \in \mathbb{R}^n \text{ with } d(m, A) < \varepsilon\}.$$

Let *power set* $P(\mathbb{R}^n)$ denote the set of all subsets of $\mathbb{R}^n$. Then we can define

Definition 12 *The* Hausdorff distance *d on the power set $P(\mathbb{R}^n)$ is defines a distance for all pairs of non-empty subsets $P, Q \subset \mathbb{R}^n$*

$$d(P, Q) := \inf_{\varepsilon > 0} \{P \subset Q_\varepsilon \text{ and } Q \subset P_\varepsilon\}$$

where Q_ε resp. P_ε are the $\varepsilon-$balls around Q resp. P. The same definition extends to subsets of any metric space.

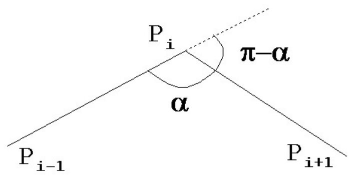
7.4 Curvature of Polygons

The most natural generalization of the curvature of smooth curves to polygons uses the rate of change of the tangent vector.

Let $\gamma = [p_0, .., p_n]$ be polygon given by a sequence of consecutive vertices p_i. Let α_i denote the inner angle at each vertex p_i:

$$\cos \alpha_i := \frac{\langle p_{i-1} - p_i, p_{i+1} - p_i \rangle}{|p_{i-1} - p_i| \, |p_{i+1} - p_i|}.$$

Definition 13 *The total curvature $\kappa(p_i)$ of γ at p_i is the defect of the vertex angle to a straight line*



$$\kappa(p_i) := \pi - \alpha_i. \tag{7.7}$$

Instead of measuring the change of the tangent vector one can look in $\mathbb{R}^2$ at the change of the normal vector. In $\mathbb{R}^3$ one looks at the change of the normal vector in the plane spanned by the two edges at each vertex.

Gauß Image of a Curve.

The normal vectors along γ are orthognal to the tangent vector along each edge. If we include the continuation of the field at vertices then the normal vectors a continuous map map

$$N : \gamma \to \mathbb{S}^1$$

which is multivalued at vertices. The length of the circular arc $N(p_i)$ is equal to the curvature $\kappa(p_i)$ of γ.
Let γ be a planar curve. Then we are able to define the sign of the vertex angle as the sign of the determinant

$$sign \det(p_i - p_{i-1}, p_{i+1} - p_i)$$

of two consecutive edges.

Theorem 14 *Let γ be a closed planar polygon with a set of vertices P. Then the sum of all curvatures divided by 2π is equal to the sum of the winding numbers of all simply connected subcurves.*

FIGURE

As a consequence, the total curvature of a simply connected domain is $\pm 2\pi$ where the sign depends on the orientation of the polygon. This fact is another justification of the definition of discrete curvature since it relates a topological invariant to the total curvature similar as for smooth curves.

Remark 15 *An alternative definition of curvature of polygons tries to account for the wish to have infinite curvature at very sharp edges using the tangens function:*

$$\kappa^L(p_i) := \tan\frac{\pi - \alpha_i}{2}. \tag{7.8}$$

Note, both discrete curvature definitions are integral curvatures solely based on the vertex angle α_i and ignore the length of adjacent edges.

7.5 Discrete Gauß Curvature

Definition 16 *The* Gauß map $g : S \to \mathbb{S}^2$ *assigns to each point p on a surface S the tip of its normal vector $n(p)$ after it was parallel translated to the origin of $\mathbb{R}^3$.*

FIGURE 7.2. The Gauß map assigns to each point $p \in S$ of a surface its normal vector $n(p) \in \mathbb{S}^2$. At edges and vertices of a polyhedral surface the image of the Gauß map is the spherical convex hull of the normal vectors of adjacent faces.

For a smooth surface, the *Gauß curvature* $K(p)$ at a point p on S, which we introduced as determinant of the Weingarten map, can also be introduced as the limiting value

$$K(p) = \lim_{\varepsilon\to 0}\frac{\operatorname{area} g(U_\varepsilon(p))}{\operatorname{area} U_\varepsilon(p)}$$

for open neighbourhoods $U_\varepsilon(p)$ of radius less than ε of p. That means, the Gauß curvature measure the metric distortion (or distortion of area) of the Gauß map from the surface S to the unit sphere.

Definition 17 *The* total Gauß curvature $K(\Omega)$ *of a domain $\Omega \subset S$ is given by the area of its spherical image including multiplicities:* $K(\Omega) = \operatorname{area} g(\Omega)$.

On a polyhedral surface, the neighbourhood of a vertex is isometric to a cone. Metrically, each cone is characterized by the total vertex angle:

Definition 18 *Let S be a polyhedral surface and $p \in S$ a vertex. Let $\{f_1,\ldots, f_m\}$ be the set of faces of* star p*, and let θ_i be the vertex angle of the face f_i at the vertex p. Then the* total vertex angle *$\theta(p)$ is given by*

$$\theta(p) = \sum_{i=1}^{m} \theta_i(p).$$

Interior points p of a face or of an open edge have a neighbourhood which is isometric to a planar Euclidean domain, and we define $\theta(p) = 2\pi$ in these cases.

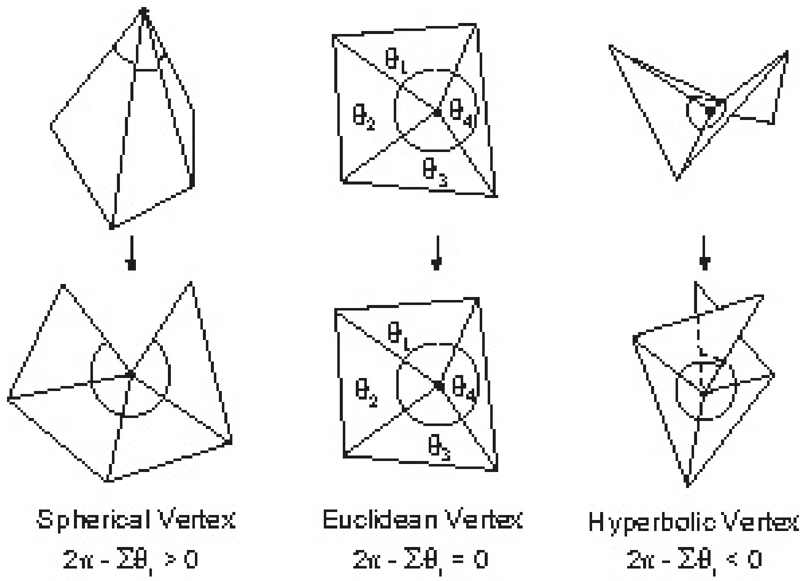


FIGURE 7.3. Classification of vertices on a polyhedral surface according to the excess of the vertex angle, and their unfolding to a planar domain.

All points of a polyhedral surface can be classified according to the sign of the *vertex angle excess* $2\pi - \theta(p)$:

Definition 19 *A vertex p of a polyhedral surface S with total vertex angle $\theta(p)$ is called* Euclidean*,* spherical*, or* hyperbolic *if its angle* *excess* $2\pi - \theta(p)$ *is* $= 0$, > 0, *or* < 0. *Similarly, interior points of a face or of an open edge are always Euclidean.*

Definition 20 *The discrete* Gauß curvature $K(p)$ *of an interior vertex* p *on a simplicial surface* S *is defined as the vertex angle excess*

$$\begin{aligned} K(p) &= 2\pi - \theta(p) \\ &= 2\pi - \sum_{i=1}^{m} \theta_i(p). \end{aligned}$$

The total Gauß curvature $K(S)$ *of a simplicial surface* S *is the sum of the Gauß curvatures of all vertices*

$$K(S) = \sum_{p \in K^{(0)}} K(p).$$

Lemma 21 *The discrete Gauß curvature* $K(p)$ *of a vertex* p *is equal to the spherical area of a polygon enclosed by the spherical convex hull of the normal vectors of triangle in the star of* p*, if the spherical polygon is convex. Formally,*

$$K(p) = \mathrm{area}_{\mathbb{S}^2}\, \mathrm{convHull}_{\mathbb{S}^2} \{n \mid n \textit{ normal vector of triangle } T \subset \mathrm{star}\, p\}$$

Example 22 *On a regular cube:* $K(p) = \frac{\pi}{2}$ *and* $K(cube) = 4\pi$.

Global topological significance of the discrete Gauß curvature:

Theorem 23 (Simplicial Gauß-Bonnet) *Let* S *be a closed simplicial surface in* $\mathbb{R}^n$ *i.e. without boundary. Then*

$$K(S) = 2\pi\chi(S).$$

Proof. Since S is closed we have

$$3f = 2e$$

where v, e, f denote the number of vertices, edges and faces of the surface S.

Let $K^{(i)}$ denote the $i-$dimensional simplices of a complex K. Then we obtain

$$\begin{aligned}
\sum_{p\in K^{(0)}} K(p) &= \sum_{p\in K^{(0)}} (2\pi - \sum_{\sigma\in \operatorname{star} p^{(2)}} \theta(p,\sigma)) \\
&= 2\pi v - \sum_{\sigma\in K^{(2)}} \sum_{p\in\sigma^{(0)}} \theta(p,\sigma) \\
&= 2\pi v - \pi f \\
&= 2\pi v - 2\pi e + 2\pi f \\
&= 2\pi\chi(S).
\end{aligned}$$

□