Prof. Dr. Konrad Polthier
Dr. Tillmann Kleiner
AG Mathematical Geometry Processing

Differential Geometry I
Winter Semester 2025/2026
Freie Universität Berlin

Exercise Sheet 10

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Submission: 12.1.2026, 8:30 AM (start of the tutorial) or 10:15 AM (start of the lecture)

Exercise 1. (4 points)
Let F and G be the functions from the Weierstraß-Enneper presentation of a conformally parametrized minimal surface f. Recall, that the inverse of the stereographic projection

$$\mathrm{st}_2\colon \mathbb{S}^2 \setminus \{(1,0,0)\} \to \mathbb{R}^2, \quad (x,y,z) \mapsto \frac{(x,y)}{1-z} \tag{1}$$

is given by

$$\mathrm{st}_2^{-1}\colon (x,y) \longmapsto \frac{(2x, 2y, x^2+y^2-1)}{x^2+y^2+1}. \tag{2}$$

Prove, that the inverse stereographic projection of G, that is, the vector field

$$N := \mathrm{st}_2^{-1}(G) = \begin{pmatrix} 2\operatorname{Re} G \\ 2\operatorname{Im} G \\ G\overline{G} - 1 \end{pmatrix} \frac{1}{G\overline{G}+1}, \tag{3}$$

defines a normal field for the surface $f(\Omega)$.[1]

Exercise 2. (8 points)
Consider the vector function $\Phi\colon \mathbb{C} \to \mathbb{C}^3$ with holomorphic entries given by

$$\Phi(z) := \begin{pmatrix} \varphi_1(z) \\ \varphi_2(z) \\ \varphi_3(z) \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sinh(z) \\ -\cosh(z) \\ -\mathrm{i} \end{pmatrix}. \tag{4}$$

i) Determine the surface parametrization defined by

$$f(u,v) = \operatorname{Re} \int_0^{z=u+\mathrm{i}v} \Phi(\xi)\,\mathrm{d}\xi. \tag{5}$$

ii) Prove, that f is a regular parametrization of a minimal surface.

iii) Verify that the parametrization f is conformal.

iv) Determine the functions F and G from the Weierstraß-Enneper presentation of f.

v) Calculate the Gauss map (= the normal field) $N\colon \mathbb{R}^2 \to \mathbb{S}^2$ from the function G, using Eq. (3).

vi) Sketch the surface parametrized by f and the normal field N.

Exercise 3. (4 points)
Let $f\colon \Omega \to \mathbb{R}^3$ and $\tilde{f}\colon \tilde{\Omega} \to \mathbb{R}^3$ be a regular surface parametrization with the corresponding Gauss maps (= normal fields) $N\colon \Omega \to \mathbb{S}^2$ and $\tilde{N}\colon \Omega \to \mathbb{S}^2$ defined by the usual cross product formula. Let $\varphi\colon \Omega \to \tilde{\Omega}$ be diffeomorphic such that $f = \tilde{f} \circ \varphi$. Prove that the relation

$$N = (\tilde{N} \circ \varphi) \cdot \operatorname{sign}(\det \mathrm{D}\varphi) \tag{6}$$

holds, where $\mathrm{D}\varphi$ is the differential of φ.

[1] Here $\operatorname{Re} z = x$, $\operatorname{Im} z = y$ resp. $\bar{z} = x - \mathrm{i}y$ denote the real part, the imaginary part resp. the complex conjugate of a complex number $z = x + \mathrm{i}y \in \mathbb{C}$.