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AG Mathematical Geometry Processing

Differential Geometry I
Winter Semester 2025/2026
Freie Universität Berlin

Exercise Sheet 9

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Submission: 5.1.2026, 8:30 AM (start of the tutorial) or 10:15 AM (start of the lecture)

Exercise 1. (4 points)
Consider a parametrized surface $M := f(\Omega)$, $f\colon \Omega \to \mathbb{R}^3$ with negative Gaussian curvature $K < 0$ everywhere. Prove the following statements:

i) At each point $p \in M$ there are two linearly independent asymptotic directions, i.e. two linearly independent tangent vectors $X, Y \in T_pM$ such that $b(X,X) = b(Y,Y) = 0$.[1]

ii) The surface M is a minimal surface if and only if the asymptotic directions in each point on M are perpendicular to each other.

Exercise 2. (4 points)
Let p be a point on a surface M. Show that the mean curvature at p is given by

$$H(p) := \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\varphi)d\varphi, \tag{1}$$

where $\kappa_n(\varphi)$ denotes the normal curvature at p in the direction v_p spanning a fixed oriented angle of φ to a fixed reference direction $v_0 \in T_pM$.

Exercise 3. (4 points)
Consider the parametrized surface $f\colon \Omega \to \mathbb{R}^3$ with

$$f(u,v) = \left(u, v, \log\left(\frac{\cos(v)}{\cos(u)}\right)\right) \qquad \Omega := \left\{(u,v) \in \mathbb{R}^2 \;\middle|\; \cos(u) \neq 0 \text{ and } \frac{\cos(v)}{\cos(u)} > 0\right\}. \tag{2}$$

i) Give a more explicit description of the domain Ω.

ii) Prove that f defines a minimal surface.

iii) Sketch the surface for $(u,v) \in \Omega \cap (-\pi/2, 3\pi/2) \times (-\pi/2, 3\pi/2)$.

Exercise 4. (4 points)
Calculate the conformally parametrized minimal surface from the Weierstraß representation with the functions $F(z) = z^2$ and $G(z) = 1/z$. Compare the resulting surface with the example from the lecture, that used $F(z) = 1$ and $G(z) = z$.

Exercise 5. (4 bonus points)
Let $f\colon \Omega \to \mathbb{R}^3$ be a regular parametrization of a surface $M := f(\Omega)$ and $u \in \Omega$ and $p = f(u) \in M$. Define the operators

$$\hat{S}_{|p}\colon T_pM \longrightarrow T_pM \qquad X \longmapsto \hat{S}_{|p}(X) := Df_{|u}(S_{|u}((Df_{|u})^{-1}(X))), \tag{3a}$$

$$\hat{b}_{|p}\colon T_pM \times T_pM \longrightarrow \mathbb{R}, \qquad (X,Y) \longmapsto \hat{b}_{|p}(X,Y) := b_{|u}((Df_{|u})^{-1}(X), (Df_{|u})^{-1}(Y)), \tag{3b}$$

where $(Df_{|u})^{-1}$ denotes the inverse of the differential of f at u, understood as a bijective linear mapping $Df_{|u}\colon T_u\Omega \to T_pM$. Further, $S_{|u}$ denotes the shape operator and $b_{|u}$ the second fundamental form at u, as obtained from the parametrization f. Prove, that the definitions of $\hat{S}_{|p}$ and $\hat{b}_{|p}$ are invariant with respect to reparametrizations, up to a multiplication with ± 1.[2]

Exercise 6. (0 points)
Enjoy your holidays and have a good start into the new year!

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[1] For $X, Y \in T_pM$ the expression $b(X,Y)$ is a short-hand for $b(Df_{|u}(v), Df_{|u}(w))$ where $u \in \Omega$ with $p = f(u)$ and $v, w \in T_u\Omega$ such that $X = Df_{|u}(v)$ and $Y = Df_{|u}(w)$. With Exercise 5 on this exercise sheet it is proved that this is well-defined, and thus, a consistent convention.

[2] That is, we get the same operators $\hat{S}_{|p}$ and $\hat{b}_{|p}$ (up to a factor ± 1) for another parametrization $\tilde{f}\colon \tilde{\Omega} \to \mathbb{R}^3$ related to f by a diffeomorphic reparametrization, which means a diffeomorphism $\varphi\colon \Omega \to \tilde{\Omega}$ such that $f = \tilde{f} \circ \varphi$.