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AG Mathematical Geometry Processing

Differential Geometry I
Winter Semester 2025/2026
Freie Universität Berlin

Exercise Sheet 5

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Submission: 24.11.2025, 8:30 AM (start of the tutorial) or 10:15 AM (start of the lecture)

Exercise 1. (6 points)
Determine the Christoffel symbols for the following surfaces:

i) $f_1 : [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi) \to \mathbb{R}^3$, $(u, v) \mapsto (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$ and

ii) $f_2 : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$, $(u, v) \mapsto ((R + r\cos(u))\cos(v), (R + r\cos(u))\sin(v), r\sin(u))$, $0 < r < R$.

Exercise 2. (6 points)
Let $c = f \circ \gamma$ be a curve parametrized by arc length which is contained in a surface patch $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$. The *Darboux frame* $\{T, B, N\}$ is defined by the relations $T(s) \coloneqq c'(s)$, $B(s) = N(s) \times T(s)$, and $N(s)$ is the surface normal at $c(s)$. The frame equations of this generalized frame are

$$\begin{pmatrix} T' \\ B' \\ N' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ B \\ N \end{pmatrix}, \tag{1}$$

where κ_g is the *geodesic curvature*, κ_n the *normal curvature*, and τ_g the *geodesic torsion*.

i) Derive the Darboux equations.

ii) Show that the normal curvature κ_n satisfies $\kappa_n = b(\gamma', \gamma')$, and that the curvature of c satisfies $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

iii) Show that if c is a Frenet curve and an *asymptotic line* (i.e. $\kappa_n(s) = 0$ for all s) then the geodesic torsion is equal to the torsion of the Frenet frame (i.e. $\tau_g(s) = \tau(s)$).

Exercise 3. (4 points)
Let $z : \Omega \to \mathbb{R}$ be a C^2-function and let $f : \Omega \to \mathbb{R}^3$, $(u, v) \mapsto (u, v, z(u, v))$ denote its graph. Show that the second fundamental form of f is given by

$$(b_{ij}) = \frac{1}{\sqrt{1 + |\nabla z|^2}} \text{hess}(z). \tag{2}$$