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AG Mathematical Geometry Processing

Differential Geometry I
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Freie Universität Berlin

Exercise Sheet 4

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Submission: 17.11.2025, 8:30 AM (start of the tutorial) or 10:15 AM (start of the lecture)

Exercise 1. (5 points)
The one parameter family of surfaces

$$\begin{aligned} f : [0, 2\pi) \times (-\infty, \infty) \times [0, \pi] &\to \mathbb{R}^3 \\ (u, v, t) &\mapsto \begin{pmatrix} \cos(t)\cos(u)\cosh(v) + \sin(t)\sin(u)\sinh(v) \\ -\cos(t)\sin(u)\cosh(v) + \sin(t)\cos(u)\sinh(v) \\ \cos(t)v + \sin(t)u \end{pmatrix} \end{aligned}$$

describes a transformation of the *catenoid* $f(-;-;0)$ into the *helicoid* $f(-;-;\pi/2)$, where f is a surface parametrized by u and v. Show that this transformation has the following properties:

i) The surface normals remain unchanged, i.e. $\frac{\partial}{\partial t}N = 0$;

ii) All surfaces $f(-;-;t)$ are isometric, i.e. $\frac{\partial}{\partial t}g = 0$;

iii) The mean curvature vanishes for all u, v and t.

Exercise 2. (7 points)
For $t \in \mathbb{R}$ consider the *tractrix* $(r, h)(t) := (1/\cosh(t), t - \tanh(t))$ and the corresponding surface of rotation[1]

$$f : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^3, (t, \varphi) \mapsto (r(t)\cos(\varphi), r(t)\sin(\varphi), h(t)).$$

i) Sketch the tractrix.

ii) Determine both principal curvatures κ_1 and κ_2.

iii) Show that its Gaussian curvature is constant.

iv) Determine its surface area.

Exercise 3. (4 points)
Let $f : \Omega \to \mathbb{R}^3$ be a parametrized surface with metric g, second fundamental form b and shape operator S. In each point $u \in \Omega$, the *third fundamental form* is a symmetric bilinear form h given by

$$h(v, w) := g(Sv, Sw) \text{ for all } v, w \in T_u\Omega.$$

Show the following equality

$$h(v, w) - 2Hb(v, w) + Kg(v, w) = 0,$$

where K denotes the Gaussian and H the mean curvature of $f(\Omega)$.

Exercise 4. (4 bonus points)
Compute the Gaussian and mean curvature for

i) $f_1 : [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi) \to \mathbb{R}^3$, $(u, v) \mapsto (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$ and

ii) $f_2 : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$, $(u, v) \mapsto ((R + r\cos(u))\cos(v), (R + r\cos(u))\sin(v), r\sin(u))$ for constants $0 < r < R$.

[1] It is also called surface of revolution and this specific example is called *tractroid* or *pseudosphere*.