

Exercise Sheet 3

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Submission: 10.11.2025, 8:30 AM (start of the tutorial) or 10:15 AM (start of the lecture)

Exercise 1.

(4 points)

For $a, b, c \in \mathbb{R}$ compute the first fundamental form of the following surfaces:

i) Ellipsoid:

$$f_1: [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto (a \sin(u) \cos(v), b \sin(u) \sin(v), c \cos(u)); \quad (1)$$

ii) Catenoid:

$$f_2: \mathbb{R} \times [0, 2\pi) \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto (a \cosh(u) \cos(v), b \cosh(u) \sin(v), cu); \quad (2)$$

and plot f_1 and f_2 .

Exercise 2.

(4 points)

Consider the sphere defined by the equation $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z - 1)^2 = 1\}$. The *stereographic projection* $\pi: S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ maps a point $p = (x, y, z)$ of S^2 minus the north pole $N = (0, 0, 2)$ to the intersection of the xy -plane with the straight line connecting N and p .

i) Show that $\pi^{-1}: \mathbb{R}^2 \rightarrow S^2$ is given by

$$\pi^{-1}(u, v) = \frac{2}{u^2 + v^2 + 4}(2u, 2v, u^2 + v^2). \quad (3)$$

ii) Compute the first fundamental form of the sphere with respect to the parametrization π^{-1} .

Exercise 3.

(4 points)

Let $\gamma(t) = (\rho(t), 0, z(t))$ be a regular parametrized curve in the xz -plane that does not meet the z -axis. γ defines a *surface of revolution* f by rotating γ around the z -axis:

$$f(t, \varphi) = (\rho(t) \cos(\varphi), \rho(t) \sin(\varphi), z(t)). \quad (4)$$

i) Show that a surface of revolution can always be parametrized so that $g_{11} = 1$, $g_{12} = g_{21} = 0$, and $g_{22} = G(t)$ where G is a function depending on t only.

ii) Show that a surface of revolution locally admits a *conformal parametrization*, that is, $g_{11} = g_{22}$ and $g_{12} = g_{21} = 0$.

Exercise 4.

(4 points)

Compute the surface area for the following parametrized surfaces:

i) Torus: For constants $0 < r < R$ set

$$f_1: [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto ((R + r \cos(u)) \cos(v), (R + r \cos(u)) \sin(v), r \sin(u)); \quad (5)$$

ii) Segment of helicoid: For $a \in \mathbb{R}_+$ set

$$f_2: (1, 2) \times [0, 2\pi) \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto (u \cos(v), u \sin(v), av). \quad (6)$$