Prof. Dr. Konrad Polthier
Dr. Eric Zimmermann
Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 1

Submission: 30.04.2024, 12:15 PM (start of lecture)

Exercise 1. (7 points)
Consider the *real projective plane* $\mathbb{R}P^2 = \mathbb{S}^2 \setminus \sim$ where

$$p \sim q \iff x = y \text{ or } x = -y \text{ for all } x, y \in \mathbb{S}^2.$$

i) Show that $\sim$ is an equivalence relation.

ii) Construct an atlas for $\mathbb{R}P^2$ consisting of exactly three charts explicitly.

Exercise 2. (4 points)
Find two smooth atlases for the real line $\mathbb{R}$ which are not smoothly compatible[1] to each other and thus yield two different smooth structures for $\mathbb{R}$. Justify your solution.

Exercise 3. (2 points)
Let $p \in \mathbb{R}^k$. A *derivation* at p is a map $X_{|p} : C^\infty(\mathbb{R}^k) \to \mathbb{R}$ that satisfies the following conditions:

* $\star$ $X_{|p}(\alpha f + \beta g) = \alpha X_{|p}(f) + \beta X_{|p}(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{R}^k)$ (linearity),
* $\star$ $X_{|p}(f \cdot g) = X_{|p}(f) \cdot g + f \cdot X_{|p}(g)$ for all $f, g \in C^\infty(\mathbb{R}^k)$ (Leibnitz rule).

i) Show $X_{|p}(f) = 0$ for f constant.

ii) Show $X_{|p}(f \cdot g) = 0$ for $f(p) = g(p) = 0$.

Exercise 4. (3 points)
Consider the following parametrization of the *hyperbolic paraboloid*

$$h : \mathbb{R}^2 \to \mathbb{R}^3, (u, v) \mapsto \begin{pmatrix} u \\ v \\ u^2 - v^2 \end{pmatrix}.$$

Further, denote with u^i, $i \in \{1, 2\}$, the standard coordinates in $\mathbb{R}^2$.

i) Use the given parametrization to derive a (single) chart φ to represent the image of $\mathbb{R}^2$ under h as a differentiable 2-manifold M.

ii) Derive coordinates (x^1, x^2) for M explicitly, where $x^i(p) = u^i(\varphi(p))$ for $p \in M$ and $i \in \{1, 2\}$.

iii) Determine the derivatives $\frac{\partial f}{\partial x^i}$, $i \in \{1, 2\}$, at a point $p \in M$ for the function $f : M \to \mathbb{R}$ which is the restriction to M of the function $(x, y, z) \mapsto z$ on $\mathbb{R}^3$ explicitly.

[1] A chart (U, φ) is *smoothly compatible* with a chart (V, ψ) if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.