Freie Universität Berlin WiSe 2024/25
Institut für Mathematik
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Differential Geometry III – Homework 11

Submission: 24. January 2025, until 8:15 am (start of the exercise class).

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1. Exercise (2 points)

Let $\mathscr{V} = \{v_0, v_1, v_2, v_3\}$ be an abstract vertex set consting of 4 distinct vertices.

i) Write down the equivalence class $[v_0, v_1, v_2]$, i.e. the set of ordered simplexes formed from the simplex $\{v_0, v_1, v_2\}$ that have the same orientation as the ordered simplex (v_0, v_1, v_2).

ii) Which of the following ordered simplices (v_0, v_1, v_2, v_3), (v_1, v_2, v_3, v_0), (v_0, v_2, v_1, v_3) and (v_2, v_1, v_4, v_0) have the same orientation?

Hint: The orientation of an ordered set of vertices flips whenever two consecutive vertices are swapped.

2. Exercise (2 points)

Consider the simplicial complex $K = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0\}, \{v_1\}, \{v_2\}, \emptyset\}$ over the abstract vertex set $\mathscr{V} = \{v_0, v_1, v_2\}$ consisting of 3 distinct vertices. Calculate the kernel and the image of the boundary operator $\partial_2 \colon C_2(K) \to C_1(K)$.

3. Exercise (2 points)

Let K be a simplicial complex over an abstract set of vertices $\mathscr{V}$. How is the dimension of the space of p-chains $C_p(K)$ calculated from K?

Hint: Use the Lemma from the last lecture (14th January).

4. Exercise (2 points)

Let K be a simplicial complex over an abstract set of vertices $\mathscr{V}$ and for any $p \in \mathbb{N}$, let $\partial_p \colon C_p(K) \to C_{p-1}(K)$ denote the boundary operator. Prove, that the composition of two boundary operators is zero, i.e.

$$\partial_{p-1} \circ \partial_p = 0 \qquad \text{for } p \in \mathbb{N},\, p \geq 2.$$

Hint: It suffices to prove that $\partial_{p-1}(\partial_p(\sigma)) = 0$ for any simplex $\sigma = (v_0, \ldots, v_p) \in K$. To get an idea of the proof for general $p \in \mathbb{N}$, try the cases $p = 1, 2$ separately first.

Total: 8