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Version: 1

Differential Geometry III – Homework 9

Submission: 10. January 2025, until 8:15 am (start of the exercise class).

This sheet contains 3 bonus points.

1. Exercise (6 points)

Let $f\colon \Omega \to \mathbb{C}$ be a complex function on a complex domain $\Omega \subset \mathbb{C}$. Define the complex function $\underline{f}\colon \overline{\Omega} \to \mathbb{C}$ by

$$\underline{f}(z) := f(\overline{z})$$

and the vector field $w\colon \Omega_r \to \mathbb{R}^2$ on the domain $\Omega_r = \{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\}$ by

$$w(x, y) := (\Re f(x + iy), \Im f(x + iy)).$$

Further, denote the partial differential operators

$$\partial_z := \tfrac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\overline{z}} := \tfrac{1}{2}(\partial_x + i\partial_y).$$

Here, for a complex number $z = x + iy$, we denote the real part as $\Re z = x$, the imaginary part as $\Im z = y$ and the complex conjugate as $\overline{z} = x - iy$.

i) Express $\partial_z f$ in terms of w, div and rot.

ii) Prove, that f is holomorphic if and only if $\partial_{\overline{z}} f = 0$.

iii) Prove, that the following three statements are equivalent:
  a) The vector field w satisfies $\operatorname{rot} w = \operatorname{div} w = 0$.
  b) The function $\overline{f}$ is holomorphic.
  c) The function $\underline{f}$ is holomorphic.

iv) Determine the complex function that corresponds to the vector field $\mathcal{J}w$.

v) Calculate the vector fields w_i associated to the functions $f_1(z) = \overline{z}$ and $f_2(z) = i\overline{z}$ on the domain $\mathbb{C}$ and $f_3(z) = 1/\overline{z}$ and $f_4(z) = i/\overline{z}$ on the domain $\Omega = \mathbb{C} \setminus \{0\}$.

vi) Sketch the vector fields w_1 and w_2 on the square domain $\{(x, y) \in \mathbb{R}^2 : |x|, |y| \leq 2\}$ and the vector fields w_3 and w_4 on the ring domain $\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$.

Please turn the page!

2. Exercise (2 points)

Liouville's theorem states that if a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is globally bounded, i.e. $\sup_{z\in\mathbb{C}} |f(z)| < \infty$, then f must be a constant. Use this theorem and the results from Exercise 1,iii) to characterize the nature of the vector fields v_3 that can appear in the Hodge-Helmholtz decomposition (1).

3. Exercise (3 points)

Any non-constant complex polynomial p can be extended to a mapping $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by setting $p(\infty) = \infty$, where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann number sphere.

i) What is the winding number of a polynomial p around the point ∞?

ii) Calculate all branching points and the corresponding winding numbers for the polynomials $p_1(z) = z^k$ with $k \in \mathbb{N}$ and for $p_2(z) = (z+1)^2(z-1)^2$.

Total: 11