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Version: 1

Differential Geometry III – Homework 8

Submission: 20. Dezember 2024, until 8:15 am by e-mail (start of the exercise class).

Let $\Omega \subset \mathbb{R}^2$ be a domain (an open set). The divergence and the rotation[I] of a vector field[II] $v\colon \Omega \to \mathbb{R}^2$ are defined as

$$\operatorname{div} v = \partial_x v_x + \partial_y v_y, \qquad \operatorname{rot} v = -\partial_y v_x + \partial_x v_y,$$

respectively, where v is written as $v(x,y) = (v_x(x,y), v_y(x,y))$ with the standard coordinates x and y. The gradient of a scalar field $f\colon \Omega \to \mathbb{R}$ is the vector field

$$\operatorname{grad} f = (\partial_x f, \partial_y f),$$

where $f = f(x,y)$. Further, define the 90°-rotation of a vector field by

$$\mathcal{J} v = (-v_y, v_x).$$

The rotated gradient $\mathcal{J} \operatorname{grad} f$ is also referred to as the cogradient. The Laplacian is denoted as $\Delta = \partial_x^2 + \partial_y^2$ and applied component wise to vector fields.

1. Exercise (3 points)

Let $f\colon \Omega \to \mathbb{R}$ and $v\colon \Omega \to \mathbb{R}^2$.

i) Verify the equations

$$\begin{aligned} \operatorname{div} \operatorname{grad} f &= \Delta f, \\ \operatorname{rot} \operatorname{grad} f &= 0, \\ \operatorname{div} \mathcal{J} \operatorname{grad} f &= 0, \\ \operatorname{rot} \mathcal{J} \operatorname{grad} f &= \Delta f. \end{aligned}$$

ii) Prove that any vector field v that is simultaneously divergence-free, i.e. $\operatorname{div} v = 0$, and rotation-free, i.e. $\operatorname{rot} v = 0$, is automatically harmonic, i.e. $\Delta v = 0$.

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[I] The rotation is also referred to as “curl” is the literature.
[II] All functions on this sheet can be assumed as infinitely differentiable (C^∞).

2. Exercise (5 points)

Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ denote the flat two-dimensional torus. The Hodge-Helmholtz decomposition theorem states that any given vector field[III] v on $\mathbb{T}^2$ can be decomposed as

$$v = v_1 + v_2 + v_3 \tag{1a}$$

with three fields v_i, $i = 1, 2, 3$ on $\mathbb{T}^2$ that satisfy

$$v_1 = \operatorname{grad} f, \qquad v_2 = \mathcal{J} \operatorname{grad} g, \qquad \operatorname{rot} v_3 = \operatorname{div} v_3 = 0. \tag{1b}$$

Here f and g are scalar fields on $\mathbb{T}^2$.

i) Let f, g be scalar fields on $\mathbb{T}^2$ and set $u_1 = \operatorname{grad} f$ and $u_2 = \mathcal{J} \operatorname{grad} g$. Prove, for $i = 1, 2$, that $u_i = 0$ if u_i is a constant vector field.

ii) Calculate a Hodge-Helmholtz decomposition for the vector field

$$v(x, y) = \begin{pmatrix} 1 + \cos x \\ \sin x + \sin y \end{pmatrix}.$$

That is, find scalar fields f, g and a vector fields v_i, $i = 1, 2, 3$ on $\mathbb{T}^2$ such that the set of Equations (1) is satisfied.

Total: 8

[III] Functions $\mathbb{T}^2 \to X$ can be represented by x, y-periodic functions $\mathbb{R}^2 \to X$. A function $f\colon \mathbb{R}^2 \to X$ is called x, y-perodic if and only if $f(x, y) = f(x + 2\pi, y) = f(x, y + 2\pi)$ for all $x, y \in \mathbb{R}$. In the above $X = \mathbb{R}$ for scalar fields and $X = \mathbb{R}^2$ for vector fields.