Freie Universität Berlin
Institut für Mathematik
Prof. Dr. K. Polthier, Dr. T. Kleiner

WiSe 2024/25

Version: 2

Differential Geometry III – Homework 7

Submission: 13. Dezember 2024, until 8:15 am (start of the exercise class).

---

1. Exercise (4 points)

The four Kepler-Poinsot polyhedra, which can be understood as generalizations of the platonic solids, are depicted in Figure 1.

i) Which of the fractional Schläfli symbols $\{5/2, 3\}$, $\{3, 5/2\}$, $\{5/2, 5\}$ and $\{5, 5/2\}$ correspond to which Kepler-Poinsot polyhedra and how are these fractional Schläfli symbols calculated?

ii) The Poinsot polyhedra $\{3, 5/2\}$ and $\{5, 5/2\}$ each correspond to a branched covering of the sphere. Determine the degree (= number of sheets) of these coverings and calculate the genus of the covering spaces using the Riemann-Hurwitz formula.

Hint: The Riemann-Hurwitz formula can be found on Slide 23 in the slides on branched covering spaces (part 1) on the webpage.

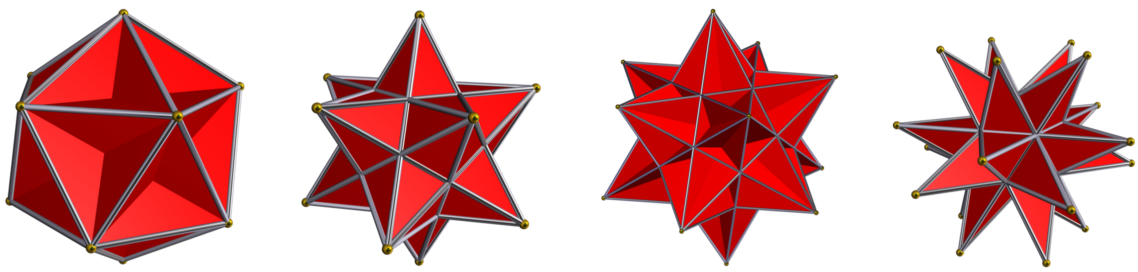

Figure 1: From left to right: The great dodecahedron, the small stellated dodecahedron, the great icosahedron and the great stellated dodecahedron. The graphics in this Figure were created using Robert Webb's software Stella from the webpage `https://www.software3d.com/Stella.php`.

Please turn the page!

2. Exercise (4 points)

The sphere inversion is the mapping defined by

$$\mathbb{R}^3 \setminus \{0\} \ni (y_1, y_2, y_3) \longmapsto \frac{(y_1, y_2, y_3)}{y_1^2 + y_2^2 + y_3^2}.$$

i) Determine the isometry of $\mathbb{S}^3 \setminus \{e_4, -e_4\}$ that corresponds to the inversion at the sphere via the stereographic projection $\widehat{\mathrm{st}}_3 \colon \mathbb{S}^3 \setminus \{e_4, -e_4\} \to \mathbb{R}^3 \setminus \{0\}$.

ii) Sketch a branched 2-sheeted covering from the surface M in Figure 2 onto the sphere such that it has a branch point with winding number 2 at every vertex of the spherical tiling that corresponds to a cube.

Hint: Find a set of 8 points that is invariant under the indicated symmetries of M. Then, using these points as vertices, draw a quad layout[I] on M and enumerate the vertices/quads corresponding to the vertices/tiles of the cubic spherical tiling.

iii) Verify by explicit calculation that the Riemann-Hurwitz formula is satisfied for the branched covering from Part ii).

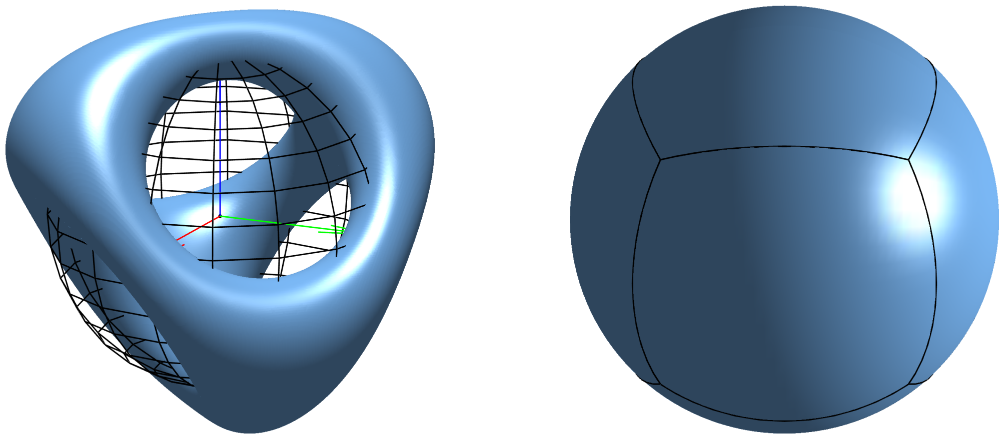

Figure 2: **Left:** A compact, closed Riemann surface M of genus 3 that has tetrahedronal symmetry and is, in addition, invariant under sphere inversion. The vectors e_1, e_2 and e_3 are indicated in red, green and blue, respectively. The unit sphere is indicated by some latitudinal and some longitudinal lines. **Right:** The spherical tiling that corresponds to the cube.

Total: 8

[I] A quad means a simply connected domain with four smooth curve segments (possibly geodesics) as boundary.