Freie Universität Berlin WiSe 2024/25
Institut für Mathematik
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Differential Geometry III – Homework 2

Submission: 8. November 2024, until 8:15 am (start of the exercise class).

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1. Exercise (6 points)

Let $\hat{st}_n : S^n \to \hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ denote the stereographic projection from the north pole $N = e_{n+1}$ (e_i denotes the i-th standard basis vector) where a point $x = (x_1, \dots, x_n, x_{n+1})$ gets mapped to a point $y = \hat{st}_n(x) = (y_1, \dots, y_n)$, i.e. $(y, 0)$ is the point where the line through N and x intersects the hyperplane $\{x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$.

i) Derive the equations for $\hat{st}_n$ and $\hat{st}_n^{-1}$.

ii) Provide a brief description and sketch for each of the following questions:
   a) What are the images of circles centered at $\pm e_3$ (lines of latitude) and circles through $\pm e_3$ (lines of longitude) under $\hat{st}_2$?
   b) What are the images of circles centered at $\pm e_1$ and circles through $\pm e_1$ under $\hat{st}_2$?
   c) How do horizontal lines $y_2 = \text{const}$ in $\hat{\mathbb{R}}^2$ look like under $\hat{st}_2^{-1}$?

iii) Let $d(x, y) = |\hat{st}_n^{-1}(x) - \hat{st}_n^{-1}(y)|$ be a metric[I] on $\hat{\mathbb{R}}^n$. Show that for $x, y \in \hat{\mathbb{R}}^n \setminus \{\infty\}$

$$d(x, \infty) = \frac{2}{\sqrt{1 + |x|^2}} \qquad \text{and} \qquad d(x, y) = \frac{2|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}.$$

Note, $|\cdot|$ denotes the Euclidean norm.

2. Exercise (2 points)

Show that the translation in S^3 given by

$$T_{zw}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(s) & -\sin(s) \\ 0 & 0 & \sin(s) & \cos(s) \end{pmatrix}$$

is conformal for all $s \in \mathbb{R}$.

Total: 8

[I] This metric is called the chordal metric. By definition $\hat{st}_n^{-1}$ is an isometry from $\hat{\mathbb{R}}^n$ with the chordal metric to S^n with the Euclidean metric.