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Scientific Visualization
Summer Semester 2023
Freie Universität Berlin

Exercise Sheet 10

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Exercise 1. (2 points) Let $v : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field. Show that $\operatorname{curl}(v) = -\operatorname{div}(Jv)$ and $\operatorname{curl}(Jv) = \operatorname{div}(v)$ with $J = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$ denoting a rotation matrix in dimension 2 by angle $\alpha \in [0, 2\pi)$. Which angle in the rotation is needed?

Exercise 2. (4 points) Let $v : \mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}$ be a vector field and $\gamma : [0,1] \to \mathbb{R}^2$, $t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$ a curve. Calculate[1] $\int_0^1 < v(\gamma(t)), \gamma'(t) > dt$. What does the result mean for the vector field v? Can you provide a potential function such that v accounts as its gradient field?

Exercise 3. (2 points) Let M be a closed simplicial surface and $v \in \Lambda^1(M)$ a differentiable vector field on M. Derive a notion[2] for $\operatorname{curl} v(x) := \int_{\operatorname{link}(e)} v$ where x lies on the edge e and is neither one of the two endpoints. Provide an example for v s.t. the curl of v vanishes at x.

Exercise 4. (8 points) Consider the flat triangulation M with vertices

$$P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } P_{k+1} = \begin{pmatrix} \cos\left(\frac{2\pi k}{6}\right) \\ \sin\left(\frac{2\pi k}{6}\right) \end{pmatrix} \text{ for } k \in \{0, \dots 5\}.$$

For the following two vector fields $v^i \in \Lambda^1(M)$ on M determine the discrete divergence and rotation at the center vertex P_0:

i) v^1 given by $v_0 = \ldots = v_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

ii) v^2 given by $v_0 = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$, $v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $v_2 = -\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$, $v_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$, $v_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_5 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$. Sketch v^2 as shown below.

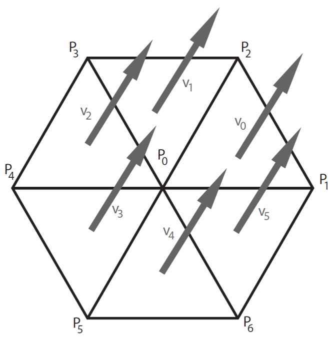

Figure 1: Not a scale model.

[1] Here we refer to the integrability of vector fields discussed in lecture 20.
[2] Similar to those we derived in the lecture for the curl of v at x being a vertex or lying inside a triangle.