Prof. Dr. Konrad Polthier
Eric Zimmermann
Version: 1

Differential Geometry I
Winter Semester 2023/2024
Freie Universität Berlin

Bonus Sheet

Submission: 20.02.2024, 12:15 PM

Note: This sheet contains 18 bonus points.

Exercise 1. (4 points)
Determine the curvature κ and torsion τ of the curve $t \mapsto (\sin(t), -t, \cos(t))$.

Exercise 2. (5 points)
For a profile curve mapping $u \mapsto (r(u), h(u))$ let

$$\begin{aligned} f : [a, b] \times [0, 2\pi) &\to \mathbb{R}^3 \\ (u, v) &\mapsto \begin{pmatrix} \cos(v) r(u) \\ \sin(v) r(u) \\ h(u) \end{pmatrix} \end{aligned}$$

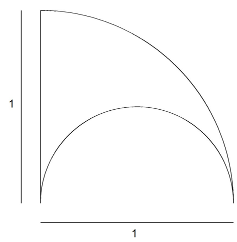
parametrize a *surface of revolution*. Determine for f the differential Df, surface normal N, metric g, second fundamental form b, shape operator S, principal curvatures κ_1, κ_2, Gaussian curvature K, and mean curvature H.

Exercise 3. (3 points)
Is there a surface $f : \Omega \to \mathbb{R}^3$ with the following first fundamental form g and shape operator S, i.e.

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2(u) \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}?$$

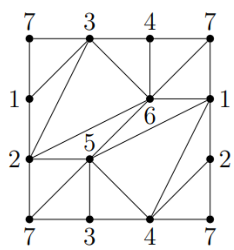
Justify your solution.

Exercise 4. (3 points)
The hyperbolic space of dimension 2 can be modeled via the upper half plane $\mathbb{H}^2 = \left\{(u, v) \in \mathbb{R}^2 \mid v > 0\right\}$ equipped with the metric $g(u, v) = \frac{1}{v^2} I_2$, with I_2 the identity matrix of dimension 2. Consider the following geodesic triangle[1]:



Determine its area by integration.

Exercise 5. (3 points)
Determine star([1,6]), link([7]), and the total Gauß curvature of the *Császár torus*[2] shown below.



[1] The triangle is enclosed by the three arcs.
[2] A model of the *Császár torus* can be found in JavaView, i.e. in File - Open - JavaView Models in the category Polytope.

13.1 $\gamma(t) = \begin{pmatrix} \sin(t) \\ -t \\ \cos(t) \end{pmatrix}$, $\gamma'(t) = \begin{pmatrix} \cos(t) \\ -1 \\ -\sin(t) \end{pmatrix}$, $|\gamma'(t)| = \sqrt{\cos^2(t) + 1 + \sin^2(t)} = \sqrt{2} \neq 1$, i.e.

not arc-length parametrized but constant speed.

Thus $\gamma(s) = \begin{pmatrix} \sin\left(\frac{s}{\sqrt{2}}\right) \\ -\frac{s}{\sqrt{2}} \\ \cos\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix}$ is an arc-length reparametrization, i.e.

$$\gamma'(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\left(\frac{s}{\sqrt{2}}\right) \\ -1 \\ -\sin\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix} \text{ and } |\gamma'(s)| = \sqrt{\frac{1}{2}\cos^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{2} + \frac{1}{2}\sin^2\left(\frac{s}{\sqrt{2}}\right)} = 1$$

Then curvature $\kappa(s) := |\gamma''(s)| = \left| \begin{matrix} -\frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right) \\ 0 \\ -\frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right) \end{matrix} \right| = \sqrt{\frac{1}{4}\sin^2(\sim) + \frac{1}{4}\cos^2(\sim)} = \frac{1}{2}$

For torsion τ we need normal $n := \frac{\gamma''(s)}{|\gamma''(s)|} = -\begin{pmatrix} \sin\left(\frac{s}{\sqrt{2}}\right) \\ 0 \\ \cos\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix}$, $n' = \begin{pmatrix} -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) \\ 0 \\ \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix}$

and binormal $b := \gamma' \times n$, i.e.

$$b(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\left(\frac{s}{\sqrt{2}}\right) \\ -1 \\ -\sin\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix} \times \begin{pmatrix} -\sin\left(\frac{s}{\sqrt{2}}\right) \\ 0 \\ -\cos\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix}.$$

Then $\tau(s) = \langle n'(s), b(s) \rangle = \left\langle \begin{pmatrix} -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) \\ 0 \\ \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix} \right\rangle = -\frac{1}{2}\cos^2(\sim) - \frac{1}{2}\sin^2(\sim)$

$= -\frac{1}{2}$

[Instead of reparametrizing to arc-length, we also could use

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} \quad \text{and} \quad \tau = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}$$

for γ Frenet curve in $\mathbb{R}^3$ (see Kühnel - Differential Geometrie)]

13.2

$$f(u,v) = \begin{pmatrix} \cos(v)\, r(u) \\ \sin(v)\, r(u) \\ h(u) \end{pmatrix}$$

$$Df = \begin{pmatrix} \underbrace{\begin{matrix} \cos(v) r'(u) \\ \sin(v) r'(u) \\ h'(u) \end{matrix}}_{\frac{\partial f}{\partial u} = f_u} & \underbrace{\begin{matrix} -\sin(v) r(u) \\ \cos(v) r(u) \\ 0 \end{matrix}}_{\frac{\partial f}{\partial v} = f_v} \end{pmatrix}, \quad \tilde{N} = f_u \times f_v = \begin{pmatrix} -\cos(v)\, r(u)\, h'(u) \\ -\sin(v)\, r(u)\, h'(u) \\ r(u) r'(u) \end{pmatrix}$$

$$|\tilde{N}| = \sqrt{r^2(u) h'(u)^2 + r^2(u) r'(u)^2} = r(u) \sqrt{h'(u)^2 + r'(u)^2}$$

$$N = \frac{\tilde{N}}{|\tilde{N}|} = \frac{1}{\sqrt{h'(u)^2 + r'(u)^2}} \begin{pmatrix} -\cos(v)\, h'(u) \\ -\sin(v)\, h'(u) \\ r'(u) \end{pmatrix}$$

$$g = \begin{pmatrix} f_u^2 & f_u f_v \\ f_u f_v & f_v^2 \end{pmatrix} = \begin{pmatrix} r'(u)^2 + h'(u)^2 & 0 \\ 0 & r(u)^2 \end{pmatrix}$$

$$f_{uu} = \begin{pmatrix} \cos(v) r''(u) \\ \sin(v) r''(u) \\ h''(u) \end{pmatrix}, \quad f_{uv} = \begin{pmatrix} -\sin(v) r'(u) \\ \cos(v) r'(u) \\ 0 \end{pmatrix}, \quad f_{vv} = \begin{pmatrix} -\cos(v) r(u) \\ -\sin(v) r(u) \\ 0 \end{pmatrix}$$

$$b = \begin{pmatrix} N f_{uu} & N f_{uv} \\ N f_{uv} & N f_{vv} \end{pmatrix} = \frac{1}{\sqrt{h'(u)^2 + r'(u)^2}} \begin{pmatrix} -r''(u) h'(u) + r'(u) h''(u) & 0 \\ 0 & r(u)\, h'(u) \end{pmatrix}$$

$$g^{-1} = \frac{1}{r(u)^2 \left(r'(u)^2 + h'(u)^2\right)} \begin{pmatrix} r(u)^2 & 0 \\ 0 & r'(u)^2 + h'(u)^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{r'(u)^2 + h'(u)^2} & 0 \\ 0 & \frac{1}{r(u)^2} \end{pmatrix}$$

$$S^t = b g^{-1} = \begin{pmatrix} \underbrace{\frac{r'h'' - r''h'}{(r'^2 + h'^2)^{3/2}}}_{\kappa_1} & 0 \\ 0 & \underbrace{\frac{h'}{r(r'^2 + h'^2)^{1/2}}}_{\kappa_2} \end{pmatrix} = S$$

$$K := \kappa_1 \kappa_2 = \det S = \frac{(r'h'' - r''h')h'}{(r'^2 + h'^2)^2\, r}$$

$$H := \frac{1}{2}(\kappa_1 + \kappa_2) = \operatorname{tr} S = \frac{1}{2}\left(\frac{r'h'' - r''h'}{(r'^2 + h'^2)^{3/2}} + \frac{h'}{r(r'^2 + h'^2)^{1/2}} \right)$$

(13.3) $g = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2(u) \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

From $S^\dagger = b g^{-1}$ we get $b = S^\dagger g = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2(u) \end{pmatrix}$.

Use notation $g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and $b = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$. Then check Mainardi Codazzi eq. (L12) and Gauß eq. (L11), i.e. simplified versions cf. exercise sheet 6.

$$\left.\begin{array}{l} \text{I)}\ L_v = \frac{E_v}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = E_v H \\ \text{II)}\ N_u = \frac{G_u}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = G_u H \end{array}\right\} \text{M.C. eq.}$$

$$\text{III)}\ K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) \quad \text{] G. eq.}$$

Check I) $L_v = \cancel{1_v}^{0} = 0 = \frac{\cancel{1_v}^{0}}{2}(1+1) = E_v H$ ✓

II) $N_u = -2\cos(u)\sin(u) = \frac{-2\cos(u)\sin(u)}{2}(1+1) = G_u H$ ✓

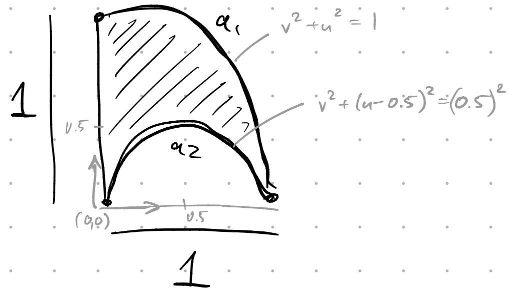
III) $K := \det S = 1 = 1 = \cancel{-}\frac{1}{\cancel{2}\sqrt{\cos^2(u)}}\left(\left(\frac{\cancel{1_v}^{0}}{\sqrt{\cos^2(u)}}\right)_v + \underbrace{\left(\frac{-\cancel{2}\cos(u)\sin(u)}{\sqrt{\cos^2(u)}}\right)_u}_{(*)}\right)$ ✓

$$\Gamma_{(*)} = \frac{(-\sin^2 + \cos^2)\sqrt{\cos^2} - (\cos\sin)\cancel{\tfrac{1}{2}}(\cos^2)^{-\frac{1}{2}}(-\cancel{2}\cos\sin)}{\cos^2}$$

$$= \frac{\sqrt{\cos^2}\,(\cancel{-\sin^2} + \cos^2 - \cancel{\sin^2})}{\cos^2} = \sqrt{\cos^2} \quad \lrcorner$$

A surface w/ given g and S exists.
(It is the sphere, $(u,v) \mapsto \begin{pmatrix} \cos(v)\cos(u) \\ \sin(v)\cos(u) \\ \sin(u) \end{pmatrix}$).

13.4 To get area of shaded triangle we subtract area beneath arc a_2 from a_1.



Similar to sheet 7 ex 4 we can express the area between a_1 and a_2 as

$$\int_0^1 \int_{\sqrt{u-u^2}}^{\sqrt{1-u^2}} \frac{1}{v^2}\,dv\,du = \int_0^1 \left[-\frac{1}{v}\right]_{\sqrt{u-u^2}}^{\sqrt{1-u^2}} du$$

$$= \int_0^1 \underbrace{\left(-\frac{1}{\sqrt{1-u^2}}\right)}_{(*)} - \underbrace{\left(-\frac{1}{\sqrt{u-u^2}}\right)}_{(\tilde{*})} du = \frac{\pi}{2}$$

$(*) = -\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$ subst. $u=\sin(x)$, $x=\sin^{-1}(u)$, $\frac{du}{dx}=\cos(x)$

$$= -\int_{0=\sin^{-1}(0)}^{\frac{\pi}{2}=\sin^{-1}(1)} \frac{1}{\sqrt{1-\sin^2(x)}}\cdot\cos(x)\,dx = -\int 1\,dx = -[x]_0^{\frac{\pi}{2}} = -\frac{\pi}{2}$$

$\sin^2+\cos^2=1$

$$(\tilde{*}) = \int_0^1 \frac{1}{\sqrt{u-u^2}}\,du$$

$$= \int_0^1 \frac{1}{\sqrt{\frac{1}{4}-\left(u-\frac{1}{2}\right)^2}}\,du$$ subst. $u-\frac{1}{2}=\frac{\sin(x)}{2}$, $x=\sin^{-1}(2u-1)$,

$u=\frac{\sin(x)+1}{2}$, $\frac{du}{dx}=\frac{\cos(x)}{2}$

$$= \int_{\frac{3\pi}{2}=\sin^{-1}(-1)}^{\frac{\pi}{2}=\sin^{-1}(1)} \frac{1}{\frac{1}{2}\sqrt{1-\sin^2(x)}}\,\frac{\cos(x)}{2}\,dx$$

$$= \int_{3\frac{\pi}{2}=a}^{\frac{\pi}{2}=b} 1\,dx \overset{\text{as } a>b}{=} -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1\,dx = [x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \pi$$

(13.5) star([1,6]) = { [1,6,7], [1,6], [1,7], [6,7], [1], [6], [7],
[1,5,6], [1,5], [5,6], [5], $\emptyset$ }

link([7]) = { [1,3], [4,6], [1,6], [2,4], [3,5], [2,5],
[1], [2], [3], [4], [5], [6], $\emptyset$ }

Total Gauß curvature Tk can be determined via discrete Gauß Bonnet (L18), i.e. $Tk = \sum_p k(p) = 2\pi\chi$.

As our torus is closed we count v, e, f, i.e.

$v = 7$, $e = 21$, $f = 14$ $\Rightarrow$ $\chi = v - e + f = 7 - 21 + 14 = 0$.

Thus $Tk = 0$.