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Version: 2

Differential Geometry I
Winter Semester 2023/2024
Freie Universität Berlin

Exercise Sheet 6

Submission: 12.12.2023, 12:15 PM (start of lecture)

Note: This sheet contains 3 bonus points.

Exercise 1. (4 points)
Let $c = f \circ \gamma$ be an arc-length parametrized curve that is contained in a surface $f : \Omega \to \mathbb{R}^3$. Prove the following statements.

i) If c'' is parallel to the normal of the surface, then c is a geodesic.

ii) The curve c is a straight line if and only if it is both an asymptotic line (i.e. $\kappa_n(s) = 0$ for all s) and a geodesic.

iii) If c is a *line of curvature* ($c'(s)$ is principal curvature direction for all s) and a geodesic, then c is a planar curve.

Exercise 2. (4 points)
Let $f : [0, \pi) \times [0, 2\pi) \to \mathbb{R}^3, (u, v) \mapsto (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))$ parametrize the unit sphere. Determine an explicit parametrization of a geodesic c with initial values $p = c(0) = (0, 0, 1)^t$ and $c'(0) = (1, 0, 0)^t \in T_p f$, and show that c satisfies the differential equations of geodesics.

Exercise 3. (4 points)
Let the first and second fundamental forms for a surface f with parameters (u, v) be given by

$$g = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} L & 0 \\ 0 & N \end{pmatrix}.$$

Show that the integrability conditions, i.e. the Codazzi-Mainardi equations simplify to

$$L_v = \frac{E_v}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = E_v \cdot H$$
$$N_u = \frac{G_u}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = G_u \cdot H$$

and the Gauß equation simplifies to

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

where K and H denote the Gaussian and mean curvature respectively.

Exercise 4. (3 points)
Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a conformal parametrization with conformal factor λ^2. Show that

$$f_{uu} + f_{vv} = 2\lambda^2 H N$$

with H the mean curvature and N the surface normal of f.

Exercise 5. (4 points)
Is there a surface $f : \Omega \to \mathbb{R}^3$ with the following first and second fundamental forms?

i) $(g_{ij})|_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(b_{ij})|_{(u,v)} = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}$;

ii) $(g_{ij})|_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2(u) \end{pmatrix}$ and $(b_{ij})|_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(u) \end{pmatrix}$.