Prof. Dr. Konrad Polthier
Eric Zimmermann
Version: 1

Differential Geometry I
Winter Semester 2023/2024
Freie Universität Berlin

Exercise Sheet 3

Submission: 14.11.2023, 12:15 PM (start of lecture)

Exercise 1. (4 points)
For $a, b, c \in \mathbb{R}$ compute the first fundamental form of the following surfaces:

i) Ellipsoid:

$$f_1 : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3, (u, v) \mapsto (a \sin(u) \cos(v), b \sin(u) \sin(v), c \cos(u))\,;$$

ii) Catenoid:

$$f_2 : \mathbb{R} \times [0, 2\pi), (u, v) \mapsto (a \cosh(u) \cos(v), b \cosh(u) \sin(v), cu)\,;$$

and plot f_1 and f_2.

Exercise 2. (4 points)
Consider the sphere defined by the equation $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z-1)^2 = 1\}$. The *stereographic projection* $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ maps a point $p = (x, y, z)$ of S^2 minus the north pole $N = (0, 0, 2)$ to the intersection of the xy-plane with the straight line connecting N and p.

i) Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$\pi^{-1}(u, v) = \frac{2}{u^2 + v^2 + 4}(2u, 2v, u^2 + v^2).$$

ii) Compute the first fundamental form of the sphere with respect to the parametrization π^{-1}.

Exercise 3. (4 points)
Let $\gamma(t) = (\rho(t), 0, z(t))$ be a regular parametrized curve in the xz-plane that does not meet the z-axis. γ defines a *surface of revolution* f by rotating γ around the z-axis:

$$f(t, \varphi) = (\rho(t) \cos(\varphi), \rho(t) \sin(\varphi), z(t))\,.$$

i) Show that a surface of revolution can always be parametrized so that $g_{11} = 1$, $g_{12} = g_{21} = 0$, and $g_{22} = G(t)$ where G is a function depending on t only.

ii) Show that a surface of revolution locally admits a *conformal parametrization* ($g_{11} = g_{22}$ and $g_{12} = g_{21}$).

Exercise 4. (4 points)
Compute the surface area for the following parametrized surfaces:

i) Torus: For constants $0 < r < R$ set

$$f_1 : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3, (u, v) = ((R + r \cos(u)) \cos(v), (R + r \cos(u)) \sin(v), r \sin(u))\,;$$

ii) Segment of helicoid: For $a \in \mathbb{R}_+$ set

$$f_2 : (1, 2) \times [0, 2\pi] \to \mathbb{R}^3, (u, v) \mapsto (u \cos(v), u \sin(v), av)\,.$$