Prof. Dr. Konrad Polthier
Eric Zimmermann
Version: 1

Differential Geometry I
Winter Semester 2023/2024
Freie Universität Berlin

Exercise Sheet 1

Submission: 31.10.2023, 12:15 PM (start of lecture)

Exercise 1. (4 points)
An *epicycloid*[1] is the path traced out by a point on a circle c which rolls around a fixed circle C. Let r and R denote positive radii of c and C, respectively. Derive a parametrization of the epicycloid and plot[2] the curve for $r = 1$ and $R = 3$.

Exercise 2. (7 points)
Show the following properties of the *Bernstein polynomials* with $n \in \mathbb{N}_0$ and $i \in [n]_0 = \{0, \ldots, n\}$:

i) $\frac{d}{dt} B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$;

ii) $B_i^n(t)$ has exactly one maximum in $[0,1]$ for $n > 0$;

iii) $B_i^n(t) = \frac{i+1}{n+1} B_{i+1}^{n+1}(t) + \frac{n+1-i}{n+1} B_i^{n+1}(t)$.

Exercise 3. (3 points)
Consider the following control points

$$P_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, P_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, P_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \mathbb{R}^2.$$

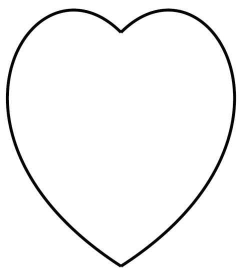
Determine $\gamma(t) = \sum_{i=0}^{2} B_i^2(t) P_i$ explicitly, transform it into the monomial basis $\left\{t^k\right\} : k \leq 3$, and sketch $\gamma(t)$. Show that γ is a regular curve.

Exercise 4. (2 points)
Let $\gamma : I \to \mathbb{R}^2$, $t \mapsto \gamma(t)$ be a regular C^2-curve (not necessarily parametrized by arc length). Its curvature is defined as

$$\kappa = \frac{\det\left(\gamma', \gamma''\right)}{\|\gamma'\|^3}.$$

Show that this definition is consistent with the curvature function given for curves parametrized by arc length, i.e. if γ is parametrized by arc length, then $\kappa = \langle \gamma'', J\gamma' \rangle$ where J denotes the rotation by $\frac{\pi}{2}$.

Exercise 5. (2 bonus points)
Design[3] a *heart shape* (see below) using two cubic Beziér curves b_1, b_2 and transform your curves b_i into the monomial basis $\left\{t^k\right\} : k \leq 3$. What are the control points?



[1]A related curve is the *cycloid*, where c rolls along a straight line. You can visualize these curves with *JavaView* (File - New - Project - Curves - Cycloid Curves).

[2]Examples for plotting parametric curves are *WolframAlpha* using the command *parametric plot* or *JavaView* (File - New - Geometry - Curves - Parametrized Curve).

[3]Vector graphics software such as *Inkscape* usually provide Beziér spline modelling or you could use the *De Casteljau algorithm* implemented in *JavaView* (File - New - Project - Curves - De Casteljau).