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Differential Geometry II – Homework 10

Submission: July 16th, 2018, 12:15 am

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1. Exercise (8 points)

1.) Recall that the compactified complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is diffeomorphic to $\mathbb{S}^2$ via *stereographic projection* $\Phi : \mathbb{S}^2 \to \hat{\mathbb{C}}$. Show that

$$\pi : \mathbb{S}^3 \subset \mathbb{C}^2 \to \mathbb{S}^2,\ (z_1, z_2) \mapsto \Phi^{-1}\left(\frac{z_1}{z_2}\right)$$

is a smooth projection whose preimage $\pi^{-1}(p)$ of each point $p \in \mathbb{S}^2$ is a circle in $\mathbb{S}^3$, a Hopf fiber, where $p = (x, y, z, t) \in \mathbb{S}^3 \subset \mathbb{R}^4 \hat{=} (z_1, z_2) = (x + iy, z + it) \in \mathbb{C}^2$.

2.) Let E denote the equator of $\mathbb{S}^2$, i.e. the preimage of the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ under Φ. Then $\pi^{-1}(E)$ is a torus in $\mathbb{S}^3$, the so-called *Clifford torus*. Show the following statements:

a) The Clifford torus is flat, i.e. its Gaussian curvature vanishes.

b) The Clifford torus is a minimal surface in $\mathbb{S}^3$.

Hint: Probably, *Hopf coordinates* on $\mathbb{S}^3$ can be useful:

$$(\theta, u, v) \in [0, \frac{\pi}{2}] \times [0, 2\pi] \times [0, \pi] \mapsto \begin{pmatrix} \cos(\theta)\cos(u+v) \\ \cos(\theta)\sin(u+v) \\ \sin(\theta)\cos(u-v) \\ \sin(\theta)\sin(u-v) \end{pmatrix}.$$

2. Exercise (4 points)

$\mathrm{SU}(n)$ denotes the *special unitary group*, i.e.

$$\mathrm{SU}(n) = \{A \in \mathbb{C}^{n,n} | A^H A = I_n \text{ and } \det(A) = 1\}.$$

Determine the Lie algebra of $\mathrm{SU}(n)$.

3. Exercise (4 points)

X, Y be two left-invariant vector fields on a Lie group G. Show that the Lie bracket $[X, Y]$ again generates a left-invariant vector field on the Lie group G.

Total: 16