Freie Universität Berlin SoSe 18
Institut für Mathematik
Prof. Dr. K. Polthier, H.-S. Lipschütz Status: May 23, 2018

Differential Geometry II – Homework 04

Submission: May 30, 2018, 12:15 am

---

1. Exercise (4 points)

Let (M, g) be a Riemannian manifold with Levi Cività connection ∇. Let R be the $(1,3)$ curvature tensor

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are arbitrary vector fields on M.

1.) Show that R is tensorial in Z, i.e. for a smooth function f, $R(X;Y)(fZ) = fR(X;Y)Z$ holds.

2.) Compute the coordinates R^l_{ijk} of R in the coordinate expression

$$R(\partial_i, \partial_j)\partial_k = \sum_l R^l_{ijk} \partial_l.$$

2. Exercise (2 points)

Let (M, g) be a Riemannian manifold and let $\gamma : I \to M$ be a unit speed geodesic. A Jacobi field J along γ with $J \perp \gamma'$ everywhere is called a *normal* Jacobi field along γ. Show that $J'' + K_{\mathbb{T}_p} J = 0$ for any normal Jacobi field if (M, g) has constant sectional curvature $K_{\mathbb{T}_p} \in \mathbb{R}$.

3. Exercise (4 points)

Find a basis of the space of Jacobi fields J_c in the follwoing cases:

1.) a straight line c in $\mathbb{R}^2$,

2.) a meridian c on $\mathbb{S}^2$.

4. Exercise (6 points)

Consider the open unit disk in $\mathbb{R}^2$ given in polar coordinates $\{(r, \varphi) \in [0,1[\times [0, 2\pi[$ with the following metric

$$g = \frac{4}{(1-r^2)^2} g_{\text{Euclid.}} = \frac{4}{(1-r^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

1.) Sketch ∂_r and ∂_φ and determine $|\partial_r|$ and $|\partial_\varphi|$.

2.) Determine $\nabla_{\partial_r}\partial_r$, $\nabla_{\partial_r}\partial_\varphi$, $\nabla_{\partial_\varphi}\partial_r$, $\nabla_{\partial_\varphi}\partial_\varphi$, and $\nabla_V W$ for $V = r\partial_r + r^2\partial_\varphi$ and $W = \varphi\partial_r + r\varphi\partial_\varphi$. Why do two of these derivatives coincide?

Total: 16