Freie Universität Berlin
Institut für Mathematik
Prof. Dr. K. Polthier, H.-S. Lipschütz

WiSe 17/18

Status: January 15, 2018

Differential Geometry I – Homework 10

Submission: January 22, 2018, 12:15 am

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1. Exercise (6 points)

Show the following calculation rules of the covariant derivative:
Let $\Omega \subset \mathbb{R}^2$ be an open set and let v, v_i, w, $w_i : \Omega \to \mathrm{T}\Omega$ be vector fields on Ω and φ, $\varphi_i : \Omega \to \mathbb{R}$ functions, $i \in \{1, 2\}$:

1.) *functional linearity in* v: $\nabla_{\varphi_1 v_1 + \varphi_2 v_2} w = \varphi_1 \nabla_{v_1} w + \varphi_2 \nabla_{v_2} w$,

2.) *product rule*: $\nabla_v(\varphi w) = (\nabla_v \varphi) w + \varphi \nabla_v w$,

3.) *compability with metric*: $\nabla_v g(w_1, w_2) = g(\nabla_v w_1, w_2) + g(w_1, \nabla_v w_2)$.

2. Exercise (4 points)

Let $f(\Omega)$ be the sphere $\mathbb{S}^2$ and let

$$c : \mathbb{R} \to \mathbb{S}^2, \ \ t \mapsto (\cos(t)\cos(\vartheta), \sin(t)\cos(\vartheta), \sin(\vartheta))$$

be a circle of latitude at fixed height $\vartheta \in (\frac{-\pi}{2}, \frac{\pi}{2})$.
Show: the covariant derivative $\nabla_{\dot{c}} \dot{c}$ vanishes if and only if $\vartheta = 0$.

3. Exercise (4 points)

Let S and S' be two parametrized surfaces tangential to a joint curve $c : I \to S \cap S'$, i.e. for each $t \in I$ the tangent spaces $\mathrm{T}_{c(t)}S$ and $\mathrm{T}_{c(t)}S'$ coincide as subspaces of $\mathrm{T}_{c(t)}\mathbb{R}^3$.
Show: for a given vector field $X_0 \in \mathrm{T}_{c(t_0)}S = \mathrm{T}_{c(t_0)}S'$, a tangent vector field $X(t)$ along c is the parallel transport of X_0 with respect to S if and only if it is the parallel transport of X_0 with respect to S'.

4. Exercise (2 points)

Let $M \subseteq \mathbb{R}^3$ be a regular surface and let $f : \Omega \to M$ be a parametrization of M.

1.) Show: if f is both a curvature line parametrization and a parametrization by arc length along every coordinate direction the surface is isometric to the plane.

2.) (2 additional points) The first part of this exercise shows that f cannot always be chosen such that f_u, f_v become an orthonormal basis of $\mathrm{T}_{f(p)}f(\Omega)$. But:

Show: for every point $p \in \Omega$, there exist a neighborhood Ω_p of p and orthonormal vector fields X, Y. I.e., there are vector fields X, Y such that X_q, Y_q is an orthonormal basis of $\mathrm{T}_q\Omega_p$ for every $q \in \Omega_p$.

Total: 16