**Freie Universität Berlin** WiSe 17/18
**Fachbereich Mathematik und Informatik**
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# Differential Geometry I – Homework 01

Submission: October 30, 2017, 12:15 am

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**1. Exercise** (7 points)

1.) Show that the *Bernstein polynomials*

$$B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j},\ j = 0, \dots, n,$$

form a basis of the real vector space of polynomials $\mathbb{P}^n$ of degree at most $n$ on $[0,1]$.

2.) Let $b(t) = \sum_{j=0}^{n} B_j^n(t) P_j$ be the *Bézier curve* of degree $n$ associated to the $n+1$ control points $P_0, \dots, P_n$. Show

$$b'(t) = n \sum_{j=0}^{n-1} (P_{j+1} - P_j) B_j^{n-1}(t)$$

and determine $b'(0)$ explicitly.

3.) Show that the Bézier curve $b(t)$ lies in the convex hull of its control points.

**2. Exercise** (4 points)

Consider the following control points

$$P_0 = \binom{1}{1}, P_1 = \binom{2}{2}, P_2 = \binom{3}{1} \in \mathbb{R}^2.$$

Determine $\gamma(t) = \sum_{j=0}^{2} B_j^2(t) P_j$ explicitly where $B_j^2(t)$ denotes the $j$-th *Bernstein polynomial* of degree 2. Show that $\gamma$ is a regular curve and determine its length for $t \in [0,1]$.

**3. Exercise** (5 points)

Let $\gamma : I \to \mathbb{R}^2$, $t \mapsto \gamma(t)$ be a regular $C^2$-curve which is not necessarily parametrized by arc length. Its curvature is defined as

$$\kappa = \frac{\det(\gamma', \gamma'')}{||\gamma'||^3}.$$

1.) Show that this definition is consistent with the curvature function given for curves parametrized by arc length: if $\gamma$ is parametrized by arc length, then $\kappa = \langle \gamma'', J\gamma' \rangle$ where $J$ denotes the rotation by $\pi$.

2.) Show that $\kappa$ is invariant under orientation preserving reparametrizations: if $\varphi : \tilde{I} \to I$ is bijective and $C^2$ with $\varphi' > 0$ and if $\tilde{\gamma} := \gamma \circ \varphi$ is the reparametrized curve then $\tilde{\kappa} = \kappa \circ \varphi$ where $\tilde{\kappa}$ denotes the curvature function as defined above for $\tilde{\gamma}$.
What happens if $\varphi$ reverses orientation, i.e. $\varphi' < 0$?

Total: 16