Freie Universität Berlin WiSe 17/18
Fachbereich Mathematik und Informatik
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Differential Geometry I – Homework 01

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1. Exercise (7 points)

1.) Show that the *Bernstein polynomials*

$$B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j},\ j = 0, \dots, n,$$

form a basis of the real vector space of polynomials $\mathbb{P}^n$ of degree at most n on $[0,1]$.

2.) Let $b(t) = \sum_{j=0}^{n} B_j^n(t) P_j$ be the *Bézier curve* of degree n associated to the $n+1$ control points $P_0, \dots, P_n$. Show

$$b'(t) = \sum_{j=0}^{n-1} (P_{j+1} - P_j) B_j^{n-1}(t)$$

and determine $b'(0)$ explicitly.

3.) Show that the Bézier curve $b(t)$ lies in the convex hull of its control points.

2. Exercise (4 points)

Consider the following control points

$$P_0 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), P_1 = \left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right), P_2 = \left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2.$$

Determine $\gamma(t) = \sum_{j=0}^{2} B_j^2(t) P_j$ explicitly where $B_j^2(t)$ denotes the j-th *Bernstein polynomial* of degree 2. Show that γ is a regular curve and determine its length for $t \in [0,1]$.

3. Exercise (5 points)

Let $\gamma : I \to \mathbb{R}^2$, $t \mapsto \gamma(t)$ be a regular C^2-curve which is not necessarily parametrized by arc length. Its curvature is defined as

$$\kappa = \frac{\det(\gamma', \gamma'')}{||\gamma'||^3}.$$

1.) Show that this definition is consistent with the curvature function given for curves parametrized by arc length: if γ is parametrized by arc length, then $\kappa = \langle \gamma'', J\gamma' \rangle$ where J denotes the rotation by π.

2.) Show that κ is invariant under orientation preserving reparametrizations: if $\varphi : \tilde{I} \to I$ is bijective and C^2 with $\varphi' > 0$ and if $\tilde{\gamma} := \gamma \circ \varphi$ is the reparametrized curve then $\tilde{\kappa} = \kappa \circ \varphi$ where $\tilde{\kappa}$ denotes the curvature function as defined above for $\tilde{\gamma}$.
What happens if φ reverses orientation, i.e. $\varphi' < 0$?

Total: 16