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Exercise Sheet 9

Online: 10.06.2015

Due: 24.06.2015, 4:00pm in the Tutorials

Exercise 9.1 (Lifting Geodesics, 3+3 Points + 3 Bonus Points)**.** Let $\widetilde{M}$ and M be two geodesically complete, connected Riemannian manifolds and let $\pi : \widetilde{M} \to M$ be a local isometry [1].

1. Prove that π has the *lifting property for geodesics*: for every geodesic $\gamma : [0,1] \to M$ in M and every point $p \in \widetilde{M}$ with $\pi(p) = \gamma(0)$ there is a unique geodesic $\widetilde{\gamma} : [0,1] \to \widetilde{M}$ such that $\pi(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma}(0) = p$.
2. Show that π is surjective. *Hint: normal coordinates.*
3. (Bonus) Conclude that π is a smooth covering map!

Exercise 9.2 (Fundamental Group on $\mathbb{RP}^2$, 4 Points)**.** By considering the covering $S^2 \to \mathbb{RP}^2$ explain why $\pi_1(\mathbb{RP}^2)$ contains a(n equivalence class of a) loop that is not contractible! Why does it become nullhomotopic if it is passed through twice? Illustrate your argument by providing a sequence of sketches of the homotopy.

[1] A map $\pi : \widetilde{M} \to M$ is a *local isometry* if for every point $p \in \widetilde{M}$ there is an open neighborhood $p \in U \subseteq \widetilde{M}$ such that $\pi|_U$ is an isometry.

In the following exercises we repeat some topics covered in earlier lectures

Exercise 9.3 (Extra 0 Points)**.** Make sure you know the following definitions, their properties and related statements by heart!

Smooth manifold, maximal atlas, smooth maps between manifolds, coordinates, partial derivatives of maps on manifolds, tangent space (3 definitions), derivations, vector fields, Matrix groups as smooth manifolds

Riemannian metric and its coordinate representation, Riemannian manifold, semi - Riemannian, isometry, Lie bracket and its coord. repr., (Riemannian/Levi-Civita) connection and its coord. repr., Christoffel symbols, parallel vector fields, parallel transport along a curve, orthonormal frames along a curve, geodesics and their initial value problem (IVP), exp-map, Riemannian normal coordinates

Contra-/covariant tensor fields, Riemann curvature tensor, (constant) sectional Curvature, Schur's theorem, Ricci tensor as contraction of RCT, scalar curvature as contraction of Ricci, polar map, geodesical completeness, local isometry, polar maps, local geodesic symmetry

Jacobi fields, Jacobi equation, Jacobi IVP, vector space of JF along a geodesic at a point, first fundamental group

Paths, homotopy, first fundamental group, simply-connectedness, covering spaces and covering maps, universal covering, deck transformations

Exercise 9.4 (Horospheres in the Poincaré Disc, 1+3+2+2 Points)**.** Once again we consider the Poincaré disc model (D, g_{-1}) given by the n-dimensional open unit ball equipped with the metric $g_{-1}\,|_p := \frac{4}{(1-\|p\|^2)^2}\delta_i^j$ (c.f exercise sheet 7).

A *horosphere* in this model is a (euclidean) sphere placed in the interior of D such that it touches ∂D in a single point, for instance the image S under the map $f(u,v) := \frac{1}{2}(\cos u \sin v, 1-\cos v, -\sin u \sin v)$ for the disc model in dimension $n = 3$.

1. Sketch the disc model containing S.
2. Compute $g_{ij} := g_{-1}(\partial_i f, \partial_j f)$, its derivatives and the Christoffel symbols.
3. Prove $\partial_u \Gamma_{12}^2 - \partial_v \Gamma_{11}^2 + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2 = 0$.
4. Conclude that S is flat. *Hint:* Gauss equation!

Considering Exercise 8.1 once again this means: intrinsically the horosphere is flat, but from a point of view of the surrounding hyperbolic space it has positive curvature.

The next exercise is a typical exam question. Try to solve it as fast as you can!

Exercise 9.5 (Exam-like Exercise, 1+2+2+2 Points).

1. Give the definition of a smooth manifold.
2. Let M and N be smooth manifolds. Show that the product $M \times N$ is a smooth manifold, too.
3. Assume M and N are equipped with metrics g_M, g_N. How is the product metric on $M \times N$ represented in local coordinates?
4. Prove that the real projective plane $\mathbb{R}\mathrm{P}^2$ is a smooth manifold.