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Differential Geometry II
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Freie Universität Berlin

Exercise Sheet 8

Online: 03.06.2015

Due: 10.06.2015, 4:00pm in the Tutorials

Exercise 8.1 (Extrinsic vs. Intrinsic Curvature, 4 Points)**.** Let (M, g) be a Riemannian manifold and $\overline{M} \subset M$ a submanifold of codimension 1 with normal field N. You may use without proof that $\overline{\nabla}_V W = \nabla_V W + b(V,W)N$ where $b(V,W) := \langle \nabla_V N, W\rangle = -\langle \nabla_V W, N\rangle$.

1. Proof that $\langle \overline{R}(V,W)X,Y\rangle = \langle R(V,W)X,Y\rangle + b(V,Y)b(W,X) - b(V,X)b(W,Y)$.
2. Deduce that for a plane $\Pi_p \subset T_p\overline{M} \subset T_pM$ it is

$$\overline{K}(\Pi_p) = K(\Pi_p) + \det b\,|_{\Pi_p}\,.$$

Exercise 8.2 (Geodesic Incompleteness, 4 Points)**.** Consider the upper half plane $\mathbb{H} := \{(x,y) : y > 0\} \subset \mathbb{R}^2$ equipped with the metric

$$g_q := \frac{1}{y^q}\delta_i^j$$

for some real number $q > 0$, $q \neq 2$ (for $q = 2$ you obtain the Poincaré half-plane model from exercise 6.2). Show that $(\mathbb{H}, g_q)$ is *not* geodesically complete!

Hint: Consider unit-speed geodesics starting at $(x_0, y_0) := (0,1)$.

Exercise 8.3 (Fundamental Groups, 2+2+2 Points)**.** Let M be a manifold, $p \in M$ a point and denote by $\pi_1(M;p)$ the first fundamental group of M at p.

1. Proof that the group multiplication on $\pi_1(M;p)$ is associative.
2. Find examples for M and p, such that
   a) $\pi_1(M;p)$ is Abelian;
   b) $\pi_1(M;p)$ is *not* Abelian.

   Justify your claims!