Prof. Dr. Konrad Polthier
Dipl.-Math. Konstantin Poelke

Differential Geometry II
Summer Semester 2015
Freie Universität Berlin

Exercise Sheet 7

Online: 27.05.2015

Due: 03.06.2015, 4:00pm in the Tutorials

Exercise 7.1 (Jacobi Fields along Radial Geodesics, 4 Points)**.** Let (M, g) be a Riemannian manifold with normal coordinates (U, x^i) around $p \in M$. For $W = \sum W^i \partial_i \in T_pM$, show that the Jacobi field along a radial geodesic γ (i. e. $\gamma(0) = p$) with $J(0) = 0$ and $\dot{J}(0) = W$ is given by $J(t) = t \sum W^i \partial_i$ for all t.

Exercise 7.2 (Connections, 2 Points)**.** Let ∇ and $\widetilde{\nabla}$ denote two connections [1] on a manifold M. Show that $\nabla - \widetilde{\nabla}$ is a $(1, 2)$-tensor field on M.

Exercise 7.3 (Ricci vs. Sectional, 4 Points)**.** Let M be a Riemannian manifold. Consider an orthonormal basis $\{E_1, \dots, E_{n-1}, X\}$ of T_pM at a point p, and denote by π_i the plane in T_pM given by the span $\langle\{E_i, X\}\rangle$. Prove that

$$\mathrm{Ric}(X, X) = \sum_{i=1}^{n-1} K(\pi_i)$$

where $K(\pi_i)$ denotes the sectional curvature.

Exercise 7.4 (Curvatures, 6 Points)**.** Consider the manifold $\mathbb{R}^n$ with the metric

$$g_{-1}\,|_p := \frac{4}{(1 - \|p\|^2)^2} \delta_i^j .$$

Compute

1. the curvature tensor R,
2. the Ricci curvature Ric,
3. the sectional curvature K
4. and the scalar curvature S

for $(\mathbb{R}^n, g_{-1})$.

[1] Recall that a connection does not need to be torsion-free nor compatible with the metric.