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Differential Geometry II
Summer Semester 2015
Freie Universität Berlin

Exercise Sheet 6

Online: 20.05.2015

Due: 27.05.2015, 4:00pm in the Tutorials

Exercise 6.1 (Normal Jacobi Fields, 2 Points)**.** Let (M, g) be a Riemannian manifold with a unit-speed geodesic $\gamma : I \to M$. A Jacobi field J along γ with $J \perp \dot{\gamma}$ everywhere is called a **normal** Jacobi field along γ. Show that $\ddot{J} + CJ = 0$ for any normal Jacobi field if (M, g) has constant sectional curvature $C \in \mathbb{R}$.

Exercise 6.2 (Jacobi Field on Poincaré Half-Plane, 4 Points)**.** Consider the upper half plane $\{(x, y) \in \mathbb{R} \times \,]0; \infty[\}$ with the metric of the **Poincaré half-plane model** for $\mathbb{H}^2$, namely $g = \frac{1}{y^2} g_{\text{eucl.}} = \frac{1}{y^2} \delta_{ij}$. Let γ be a constant-speed parametrisation of $\{(x_0, y) \,|\, y \in \,]0; \infty[\}$ for constant x_0.

(i) Sketch the situation and show that γ is a geodesic.

(ii) Show that $J = \partial_x$ is a Jacobi field along γ.

Exercise 6.3 (Normal Coordinate Systems, 4 Points)**.** Let (M, g) be a Riemannian manifold, $p \in M$ and $E_1, \ldots, E_n$ an orthonormal basis of $T_p M$. This basis induces an isomorphism $E : \mathbb{R}^n \to T_p M$, $(x^1, \ldots, x^n) \mapsto \sum_i x^i E_i$. For a small neighbourhood U of p in M on which $\exp_p$ is bijective, $x := E^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n$ is called a **normal coordinate system** for M at p.

The coordinates of p are $(0, \ldots, 0)$, and $g_{ij} = \delta_{ij}$ at p (why?), but not neccessarily anywhere else. Show the following additional properties of normal coordinates:

(i) For any $V = \sum V^i \partial_i \in T_p M$, the geodesic γ starting at p with initial velocity V is given by $x(\gamma(t)) = (tV^1, \ldots, tV^n)$.

(ii) The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.

Hint: Start with the Christoffel symbols and recall the defining equation for geodesics in coordinates...