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Differential Geometry II
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Freie Universität Berlin

Exercise Sheet 5

Online: 13.05.2015

Due: 20.05.2015, 4:00pm in the Tutorials

Exercise 5.1 (Curvature Tensor I, 4 Points)**.** Let (M, g) be a Riemannian manifold with Levi Cività connection ∇. Let R be the $\binom{1}{3}$ curvature tensor

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

(i) Show that R is indeed tensorial in Z, i. e. $R(X,Y)(fZ) = f\,R(X,Y)Z$ for a C^∞ function f.

(ii) Compute the coordinates ${R_{ijk}}^\ell$ of R in the coordinate expression $R(\partial_i, \partial_j)\partial_k = \sum_\ell {R_{ijk}}^\ell \partial_\ell$.

Exercise 5.2 (Curvature Tensor II, 2 Points)**.** Prove that for a fixed vector field X, the covariant derivative $\nabla_X R$ of the curvature tensor is again a $\binom{1}{3}$ tensor given by

$$(\nabla_X R)(U,V)W = \nabla_X(R(U,V)W) - R(\nabla_X U, V)W - R(U, \nabla_X V)W - R(U,V)\nabla_X W.$$

Exercise 5.3 (Jacobi Fields, 2+2 Points)**.** Find a basis for the space J_c of Jacobi fields along...

1. ... a straight line c in $\mathbb{R}^2$;
2. ... a meridian c on S^2.

Exercise 5.4 (Poincaré Disc, 2+4 Points)**.** Consider the open unit disk in $\mathbb{R}^2$, given in polar coordiates $\{(r,\varphi) \in [0;1[\times[0;2\pi[\}$, but instead of the usual Euclidean metric $g_{\text{eucl.}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & r^2 \end{smallmatrix}\right)$, we consider the **Poincaré disk model** for $\mathbb{H}^2$, that is

$$g = \frac{4}{(1-r^2)^2} g_{\text{eucl.}} = \frac{4}{(1-r^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

(i) Sketch ∂_r and ∂_φ and compute $|\partial_r|$ and $|\partial_\varphi|$.

(ii) Compute $\nabla_{\partial_r}\partial_r$, $\nabla_{\partial_r}\partial_\varphi$, $\nabla_{\partial_\varphi}\partial_r$, $\nabla_{\partial_\varphi}\partial_\varphi$ and $\nabla_V W$ for $V = r\partial_r + r^2\partial_\varphi$ and $W = \varphi\partial_r + r\varphi\partial_\varphi$. Two of these derivatives coincide. Why?