Prof. Dr. Konrad Polthier
Dipl.-Math. Konstantin Poelke

Differential Geometry II
Summer Semester 2015
Freie Universität Berlin

Exercise Sheet 10

Online: 24.06.2015

Due: 01.07.2015, 4:00pm in the Tutorials

Exercise 10.1 (Stereographic Projection of Hypercube, 2 Points)**.** Sketch the image of the four-dimensional (skeleton of the) hypercube under stereographic projection to $\mathbb{R}^3$.

Exercise 10.2 (Quaternions and Matrices, 2 Points)**.** Consider the *Pauli matrices*[1]

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof that the $\mathbb{R}$-linear span $\langle\{\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3\}\rangle_{\mathbb{R}}$ is isomorphic to the quaternion $\mathbb{R}$-algebra $\mathbb{H}$.

Exercise 10.3 (Hopf Fibration, 2 Points)**.** Recall that the compactified complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is diffeomorphic to the sphere S^2 via stereographic projection $\Phi : S^2 \to \hat{\mathbb{C}}$.

Verify that the map

$$\pi : S^3 \subset \mathbb{C}^2 \to S^2, \ (z_1, z_2) \mapsto \Phi^{-1}(z_1/z_2)$$

is a smooth projection whose preimage [2] $\pi^{-1}(p)$ is a circle in S^3.

Exercise 10.4 (Clifford Torus, 2+4 Points)**.** Let E denote the equator of S^2, i.e. the preimage of the unit circle $S^1 \subset \mathbb{C}$ under Φ. Then $\pi^{-1}(E)$ is a torus in S^3, the so-called *Clifford torus*. Prove that the Clifford torus ...

1. ... is flat, i.e. has vanishing Gauss curvature;
2. ... is a minimal surface in S^3.

Hint: It might be useful to use *Hopf coordinates* on S^3:

$$(u, v, \vartheta) \mapsto \begin{pmatrix} \cos u \sin\vartheta \\ \sin u \sin\vartheta \\ \cos v \cos\vartheta \\ \sin v \cos\vartheta \end{pmatrix} \text{ with } u, v \in [0, 2\pi], \ \vartheta \in [0, \pi/2].$$

[1] Wolfgang Pauli, 25.04.1900 - 15.12.1958, Austrian physicist

[2] The symbol $\infty \in \hat{\mathbb{C}}$ (i.e. the north pole on S^2) corresponds to the point $z_1/0$.