Prof. Dr. Konrad Polthier
Dipl.-Math. Konstantin Poelke
Dipl.-Math. Thomas Gust

Differential Geometry I
Winter Semester 2014/15
Freie Universität Berlin

Exercise Sheet 5

Online: 10.11.2014

Due: 19.11.2014, 4:00pm

Four points for each exercise!

Exercise 5.1 (Darboux Frames)**.** Let c be an arc length parametrized curve that is contained in a surface $f : U \to \mathbb{R}^3$. The Darboux frame $\{E_1, E_2, E_3\}$ is defined by the relations $E_1(s) := c'(s)$, $E_2(s) = E_3 \times E_1(s)$ and $E_3(s) = N(c(s))$ where N is the unit normal of the surface. The frame equations of this generalized frame are

$$\frac{d}{ds}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_N \\ -\kappa_g & 0 & \tau_g \\ -\kappa_N & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

where κ_g is the *geodesic curvature*, κ_N the *normal curvature*, and τ_g the *geodesic torsion.*

1. Show that the normal curvature satisfies $\kappa_N = II(c', c')$, and in the case that c is a Frenet curve its curvature satisfies $\kappa^2 = \kappa_g^2 + \kappa_N^2$.
2. Show that c is a *line of curvature*[1] if and only if the geodesic torsion vanishes, i.e. $\tau_g(s) = 0$ for all s.
3. Show that if c is a Frenet curve and an *asymptotic line*[2], then the geodesic torsion equals the torsion of the Frenet frame, i.e. $\tau_g(s) = \tau(s)$.

Exercise 5.2 (Ruled Surfaces I)**.** Let γ be a curve contained in a surface f with surface normal N and let $r(u, v) := \gamma(u) + v \cdot N(\gamma(u))$ denote the *ruled surface defined by γ and the vector field N.* Show that the following two statements are equivalent:

1. γ is a line of curvature.
2. The ruled surface r has vanishing Gauss curvature everywhere.

Exercise 5.3 (Averaging Property of Mean Curvature)**.** Let p be a point on a surface. Show that the mean curvature at p is given by

$$H(p) := \frac{1}{2\pi} \int_0^{2\pi} \kappa_N(\varphi) d\varphi,$$

where $\kappa_N(\varphi)$ denotes the normal curvature at p in the direction v_φ spanning an angle of φ to a fixed reference direction $v_0 \in T_p f$.

---

[1] c is a *line of curvature* if $c'(s)$ is a principal curvature direction for all s

[2] c is an *asymptotic line* if $II(c'(s), c'(s)) = 0$ for all s

Exercise 5.4 (Graph Surfaces)**.** Let $z : U \to \mathbb{R}$ be a C^2-function and let $f : U \to \mathbb{R}^3$, $(u, v) \mapsto (u, v, z(u, v))$ denote its graph. Show that the second fundamental form of f is given by

$$(h_{ij}) = \frac{1}{\sqrt{1 + \|\nabla z\|^2}} \operatorname{hess}(z).$$