

Exercise Sheet 4

Online: 03.11.2014

Due: 12.11.2014, 4:00pm

Four points for each exercise!

Exercise 4.1 (Catenoid – Helicoid). The one parameter family of surfaces

$$f : [0, 2\pi] \times (-\infty, \infty) \times [0, \pi] \rightarrow \mathbb{R}^3$$
$$f(u, v, t) := \begin{pmatrix} \cos(t) \cos(u) \cosh(v) + \sin(t) \sin(u) \sinh(v) \\ -\cos(t) \sin(u) \cosh(v) + \sin(t) \cos(u) \sinh(v) \\ \cos(t)v + \sin(t)u \end{pmatrix}$$

describes a transformation of the *catenoid* $f(-; -; 0)$ into the *helicoid* $f(-; -; \pi/2)$. Show that this transformation has the following properties:

1. The surface normals remain unchanged, i.e. $\frac{\partial}{\partial t} N = 0$;
2. All surfaces $f(-; -; t)$ are isometric, i.e. $\frac{\partial}{\partial t} g = 0$;
3. The mean curvature vanishes for all u, v and t .

Exercise 4.2 (Fundamental Forms). Let $f : U \rightarrow \mathbb{R}^3$ be a parametrized surface with shape operator L and Gauss map N . For every point $p \in U$ the *third fundamental form* is the symmetric bilinear form defined by $III(X, Y) := \langle LX, LY \rangle$ for all $X, Y \in T_p f$. Furthermore we write $I(X, Y) := \langle X, Y \rangle$ and $II(X, Y) := I(LX, Y)$ for the first and second fundamental form of f . Prove the equality

$$III(X, Y) - 2H \cdot II(X, Y) + K \cdot I(X, Y) = 0,$$

where K denotes the Gaussian and H the mean curvature of the surface.

Exercise 4.3 (Parallel Surface I). Let f be a parametrized surface with unit normal N . We define its *parallel surface at distance* $\epsilon > 0$ by

$$f_\epsilon(u, v) := f(u, v) + \epsilon N(u, v).$$

Prove that for the area elements dA and dA_ϵ the following relation holds:

$$dA_\epsilon = (1 - 2H\epsilon + K\epsilon^2)dA,$$

where K and H denote Gaussian and mean curvature of f .

Exercise 4.4 (Curvatures). Compute the Gauss and mean curvature for...

1. the *sphere*: $(u, v) \mapsto (\cos u \cos v, \cos u \sin v, \sin u)$;
2. the *torus*: $(u, v) \mapsto ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u)$ for constants $0 < r < R$ (cf. exercise sheet 3).