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Differential Geometry I
Winter Semester 2014/15
Freie Universität Berlin

Exercise Sheet 2

Online: 20.10.2014

Due: 29.10.2014, 4:00pm

Four points for each exercise!

Exercise 2.1 (Helix). The map

$$\mathrm{helix}_{r,a} : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto \begin{pmatrix} r \cdot \cos t \\ r \cdot \sin t \\ at \end{pmatrix}$$

parameterizes a *helix* with radius r and slant a. For fixed r and a find an arc length parameterization of the helix and compute its curvature and torsion.

Exercise 2.2 (Curvature). Let $\gamma : I \to \mathbb{R}^2$, $t \mapsto \gamma(t)$ be a regular, planar C^2-curve, which is not necessarily parametrized by arc length.

We define its curvature by

$$\kappa = \frac{\det(\gamma', \gamma'')}{\|\gamma'\|^3}.$$

1. Show that this definition is consistent with the curvature function given for arc length parametrized curves: if γ is arc length parametrized, then $\kappa = \langle \gamma'', J\gamma' \rangle$.
2. Show that κ is invariant under orientation-preserving reparametrizations: if $\alpha : \tilde{I} \to I$ is bijective and C^2 with $\alpha' > 0$ and if $\tilde{\gamma} := \gamma \circ \alpha$ is the reparametrized curve, then it is $\tilde{\kappa} = \kappa \circ \alpha$, where $\tilde{\kappa}$ denotes the curvature function as defined above for the curve $\tilde{\gamma}$. What happens if α reverses the orientation ($\alpha' < 0$)?

Definition (Frenet curve, Frenet frame). A C^{n-1}-curve $\gamma : I \to \mathbb{R}^n$ parametrized by arc length is a *Frenet curve* if the vectors $\gamma'(t), \dots, \gamma^{(n-1)}(t)$ are linearly independent for each $t \in I$. The *Frenet frame* $\{b_1(t), \dots, b_n(t)\}$ for a Frenet curve γ is the frame obtained by Gram-Schmidt orthonormalization as follows:

$$b_1 := \frac{\gamma'}{\|\gamma'\|} = \gamma', \; b_2 := \frac{\gamma'' - \langle \gamma'', b_1 \rangle b_1}{\|\gamma'' - \langle \gamma'', b_1 \rangle b_1\|}, \; \dots, b_{n-1} := \frac{\gamma^{(n-1)} - \sum_{i=1}^{n-2} \langle \gamma^{(n-1)}, b_i \rangle b_i}{\left\| \gamma^{(n-1)} - \sum_{i=1}^{n-2} \langle \gamma^{(n-1)}, b_i \rangle b_i \right\|}$$

and $b_n(t)$ being the unique vector of unit length, such that $\{b_1(t), \dots, b_n(t)\}$ form a positively oriented ($\det(b_1, \dots, b_n) = 1$) basis of $\mathbb{R}^n$ at each $t \in I$. If $n = 2$ or $n = 3$ we also write $T := b_1$, $N := b_2$, $B := T \times N = b_3$.

Exercise 2.3 (Evolute)**.** Let $\gamma : I \to \mathbb{R}^2$ be a planar Frenet curve with Frenet frame $\{T(t), N(t)\}$. Its *evolute* is the curve η defined by

$$\eta(t) := \gamma(t) + \frac{1}{\kappa(t)} N(t).$$

Compute and plot the evolute of the cycloid

$$c : (0, 2\pi) \to \mathbb{R}^2, \ t \mapsto (t - \sin t, 1 - \cos t)$$

and interpret your result.

Exercise 2.4 (Parallel Frame)**.** Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve with Frenet frame $\{T(s), N(s), B(s)\}$. For a smooth function φ the one-parameter family of rotations

$$\begin{pmatrix} \widetilde{N}(s) \\ \widetilde{B}(s) \end{pmatrix} = \begin{pmatrix} \cos\varphi(s) & -\sin\varphi(s) \\ \sin\varphi(s) & \cos\varphi(s) \end{pmatrix} \begin{pmatrix} N(s) \\ B(s) \end{pmatrix}$$

generates a new orthonormal frame $\{T(s), \widetilde{N}(s), \widetilde{B}(s)\}$.

1. Compute the torsion
$$\widetilde{\tau}(s) = \left\langle \widetilde{N}'(s), \widetilde{B}(s) \right\rangle$$
of the new frame $\{T(s), \widetilde{N}(s), \widetilde{B}(s)\}$.
2. Find a function φ such that the frame $\{T(s), \widetilde{N}(s), \widetilde{B}(s)\}$ is torsion free, i.e. $\widetilde{\tau}(s) = 0$ for all s. Such a frame is called a *parallel frame.*
3. Find a parallel frame for the arc length parametrized helix from exercise 2.1.