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Differential Geometry I
Winter Semester 2014/15
Freie Universität Berlin

Exercise Sheet 12

Online: 14.01.2015

Due: 21.01.2015, 4:00pm

Four points for each exercise!

Exercise 12.1 (Parallel Vector Fields)**.** Let X be a vector field along a geodesic γ. Show that X is parallel along γ if and only if it has constant length and makes a constant angle with $\dot{\gamma}$.

Exercise 12.2 (Gauss Curvature in Orthogonal Coordinates)**.** Let $f : U \to \mathbb{R}^3$ be an orthogonally parametrized surface, i.e. $g_{ij} = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$ is a diagonal matrix. Prove that the Gauss curvature is given by

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

Hint: You can use the Gauss equation for integrability, derive a formula for $\det(h_{ij})$ from this equation and compute the Christoffel symbols...

Exercise 12.3 (Angles between Vector Fields)**.** Prove the following lemma from the lecture:

Let X, Y be two vector fields of unit length along a curve $c : I \to S \subset \mathbb{R}^3$ contained in a surface, and let $[\nabla_{c'} X]$ denote the projection of $\nabla_{c'} X$ onto $N \times X$, where N denotes the normal of S. Then

$$[\nabla_{c'} X] - [\nabla_{c'} Y] = \varphi'$$

where φ is a differentiable function measuring the angle between X and Y (extend X to an orthonormal frame $\{X, \bar{X}\}$ and write $Y = aX + b\bar{X}$. Then φ is the function defined by a, b as in the lecture).

Exercise 12.4 (Gauss-Bonnet)**.** Prove the following facts using Gauss-Bonnet!

1. The sum of the angles of a euclidean triangle is π.
2. For a simple, closed planar curve c it is $\int_c \kappa_g = 2\pi$.
3. On a simply connected surface S with negative Gaussian curvature two geodesics emanating from a point $p \in S$ cannot meet again in a second point $q \in S$.
4. A surface which is homeomorphic to a torus always has a point with vanishing Gaussian curvature.