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Differential Geometry I
Winter Semester 2014/15
Freie Universität Berlin

Exercise Sheet 11

Online: 05.01.2015

Due: 12.01.2015, 4:00pm

Four points for each exercise!

Exercise 11.1 (The $\mathrm{SL}(2,\mathbb{R})$-action on $\mathbb{H}^2$)**.** Let

$$\mathrm{SL}(2,\mathbb{R}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \text{ with } ad - bc = 1\}$$

denote the real special linear group. Each matrix $A \in \mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H}^2 := \{z \in \mathbb{C} \cong \mathbb{R}^2 : \operatorname{Im} z > 0\}$ via the induced transformation

$$\varphi_A : z \mapsto \varphi_A(z) := \frac{az+b}{cz+d}.$$

Prove the following statements:

1. $\operatorname{Im} \varphi_A(z) > 0$ for $z \in \mathbb{H}^2$, so $\varphi_A : \mathbb{H}^2 \to \mathbb{H}^2$ is well-defined;
2. given two matrices $A, B \in \mathrm{SL}(2,\mathbb{R})$, it is $\varphi_{BA} = \varphi_B \circ \varphi_A$;
3. φ_A is a bijection with inverse $\varphi_A^{-1} = \varphi_{A^{-1}}$.

Exercise 11.2 (Area in Poincaré Disc)**.** Let $\mathbb{D}$ denote the Poincaré disc model of the hyperbolic space. Compute the area of a circle with radius $r < 1$, centered at the origin of $\mathbb{D}$.

Exercise 11.3 (Local Orthonormal Frames)**.** Let $f : \Omega \to M$, $M \subset \mathbb{R}^3$, be a regular surface.

(i) Show that if f is both a curvature line parametrisation and an arclength parametrisation along every coordinate direction, the surface has to be isometric to the plane.

(ii) Part *(i)* shows that f cannot always be chosen such that f_u, f_v become an orthonormal basis of $T_p\Omega$. But: Show that around every point $p \in \Omega$, one can find a small neighbourhood Ω' and an **orthonormal frame** X, Y, that means: vectorfields $X, Y \in \Gamma(T\Omega')$ such that X_q, Y_q is a g-orthonormal basis of $T_q\Omega$ for every $q \in \Omega'$.

Exercise 11.4 (Parallel Transport)**.** Let S and S' be two parametrized surfaces touching at a joint curve $c : I \to S \cap S'$, i.e. for each $t \in I$ the tangent spaces $T_{c(t)}S$ and $T_{c(t)}S'$ coincide (as subspaces of $T_{c(t)}\mathbb{R}^3$). Show that for a given vector $X_0 \in T_{c(t_0)}S = T_{c(t_0)}S'$ a tangent vector field $X(t)$ along c is the parallel transport of X_0 with respect to S if and only if it is the parallel transport of X_0 with respect to S'.