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Differential Geometry I
Winter Semester 2014/15
Freie Universität Berlin

Exercise Sheet 1

Online: 14.10.2014

Due: 22.10.2014, 4:00pm

Four points for each exercise!

Exercise 1.1 (Epicycloids)**.** An *epicycloid* is the path traced out by a point on a circle c which rolls around a fixed circle C. Let r and R denote the radii of c and C, respectively.

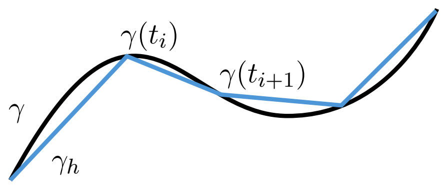
Derive a parametrization of the epicycloid and plot the curve for $r = 1$ and $R = 3$. [1]

Hint: A related curve is the *cycloid*, where c rolls along a straight line. You can visualize these curves with *JavaView* (`http://www.javaview.de`) by starting a new project: `File > New > Project > Curves > Cycloid Curves`. There is also a demo applet available online: `http://www.javaview.de/demo/PaCycloid.html`

Exercise 1.2 (Approximation)**.** Let $\gamma : [a, b] \to \mathbb{R}^n$ be a regular curve and $a = t_0 < \cdots < t_k = b$ a subdivision of the interval with mesh size $h := \max_i |t_{i+1} - t_i|$. This subdivision defines a unique interpolating approximation polygon γ_h with linear segments $\overline{\gamma(t_i)\gamma(t_{i+1})}$ (see figure below).

Prove that $L(\gamma_h) \to L(\gamma)$ for any sequence of such polygons with $h \to 0$ (and therefore $k \to \infty$).

Hint: It is enough to show that $L(\gamma_h)$ converges to a Riemann sum $\sum_i |\gamma'(s_i)|(t_{i+1} - t_i)$ with $t_i \leq s_i \leq t_{i+1}$ for the integral $\int_a^b |\gamma'(t)|dt$ (why?). To this end the keywords *mean value theorem* and *uniformly continuous* might render useful...



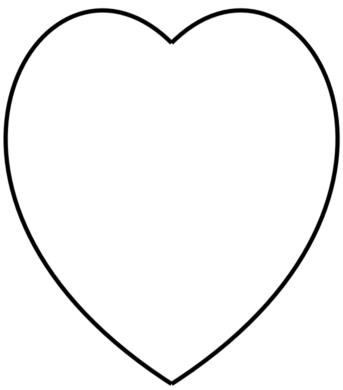
[1] Useful plotting tools are e.g. *gnuplot* (`http://www.gnuplot.info/`) or *KmPlot*, both pre-installed on the machines in the PC pools, room 030 or room 005 in the math building. *JavaView* can plot parametrized curves via `File > New > Project > Curves > Parametrized Curves`. *WolframAlpha* (`http://www.wolframalpha.com/`) is available as an online service. *Mathematica*, *Matlab* etc. certainly possess rich plotting functionality, too.

Exercise 1.3 (Bernstein and Beziér)**.** Let P_i, $i = 0, \dots, d$ be $d+1$ control points in $\mathbb{R}^n$. We denote by $B_i^d(t)$ the $d+1$ Bernstein polynomials of degree d and by $b(t) := \sum_{i=0}^d B_i^d(t) P_i$ the Beziér curve corresponding to the given control points. Prove the following facts!

1. Recursion: $B_i^d(t) = tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t)$;
2. Partition of unity: $\sum_{i=0}^d B_i^d(t) \equiv 1$;
3. $\frac{d}{dt} B_i^d(t) = d(B_{i-1}^{d-1}(t) - B_i^{d-1}(t))$;
4. $\frac{d}{dt} b(t) = d \sum_{i=0}^{d-1} (P_{i+1} - P_i) B_i^{d-1}(t)$.

Conclude that the tangents of b at $b(0) = P_0$ and $b(1) = P_d$ are parallel to the initial and ending edge segments, respectively, of the control polygon.

Exercise 1.4 (Beziér Curves, 2 Bonus Points)**.** Design a *heart shape* (see below) using two cubic Beziér curves b_1, b_2 and transform your curves b_i into the monomial basis $\{t^k : k \leq 3\}$. What are the control points? [2]



[2]Typical vector graphics software such as *Inkscape* (`www.inkscape.org`) or *Adobe Illustrator* usually provide Beziér spline modelling, however sometimes restricted to Beziér patches of degree 3.