On the Convergence of Metric and Geometric Properties of Polyhedral Surfaces *

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Abstract

We provide conditions for convergence of polyhedral surfaces and their discrete geometric properties to smooth surfaces embedded in $\mathbb{R}^3$. The notion of *totally normal convergence* is shown to be equivalent to the convergence of either one of the following: surface area, intrinsic metric, and Laplace-Beltrami operators. We further show that totally normal convergence implies convergence results for shortest geodesics, mean curvature, and solutions to the Dirichlet problem. This work provides the justification for a discrete theory of differential geometric operators defined on polyhedral surfaces based on a variational formulation.

1 Introduction

Discrete differential geometry of polyhedral surfaces studies discrete analogous of smooth differential geometric concepts. It is a theory *sui iuris* where polyhedral surfaces and discrete operators take the place of smooth ones, relying solely on the information inherent to a discrete mesh. Many properties of smooth manifolds remain valid in a purely polyhedral setting, for example, the discrete Gauß-Bonnet theorem which relates discrete Gauß curvature, discrete geodesic curvature and surface topology, closely resembles the continuous case. Polyhedral meshes are beginning to show major applications in areas such as computational mechanics and computer graphics, where discrete curvature and discrete differential operators provide the machinery for numerical simulations.

There exists a long history of rigorous definitions for discrete differential geometry. Alexandrov [1] and Reshetnyak [20] developed a theory of manifolds of bounded curvature. Thurston [23] and Schramm [21] used circle packings to approximate smooth holomorphic maps and prove a discrete Riemann mapping theorem. Federer [8] and Fu [9] developed geometric measure theory, Cheeger, Müller and Schrader [4] use Lipschitz-Killing curvatures, and Morvan and coworkers [6] [14] [15] recently used the theory of normal cycles to calculate a discrete shape operator. Combinatorial approaches were introduced by Mercat [13] and Bobenko et.al. [2] to discretize the underlying conformal structure of Riemann surfaces.

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Our methodology is to employ a *variational approach*, where energy functionals on polyhedral surfaces give rise to discrete curvature and differential operators. As a subset of Euclidean space, polyhedral surfaces carry an induced cone structure, and finite-dimensional function spaces over polyhedral surfaces arise naturally as subspaces of Sobolev spaces. Dziuk [7] was the first to use the discrete variational approach for studying solutions to the Dirichlet problem directly on polyhedral surfaces instead of planar parameter domains. The variational view provides a variety of discrete concepts: from differential-geometric operators such as gradient, divergence, and Laplace-Beltrami operator over a distributional interpretation of mean curvature to a discrete notion of geodesics (see eg. [16] [17] [18]). The variational approach has led to the first numerical construction of compact constant mean curvature surfaces of genus greater than one [11].

However, albeit its popularity, an essential piece of justification for this discrete approach has been missing: a proof that the classical smooth case arises as the limit of the discrete theory. The current paper, to a large part, provides this missing link.

We consider the following question: If a sequence of triangulated polyhedral surfaces embedded in Euclidian 3-space converges to a smooth surface in Hausdorff distance; under which conditions do metric and geometric properties such as intrinsic distance, area, mean curvature, geodesics, and Laplace-Beltrami operators converge, too? The Schwarz lantern [22] is an informative counterexample to convergence of surface area: here convergence fails since the normal fields of the approximating sequence of polyhedra diverge. Morvan [15] showed that convergence of the normal fields implies convergence of surface area. We generalize this result considerably.

In Theorem 2 we prove the following convergence result: if a sequence of polyhedral surfaces M_τ converges to a smooth surface M in Hausdorff distance then the following conditions are equivalent:

i convergence of normal fields,

ii convergence of metric tensors,

iii convergence of area,

iv convergence of Laplace-Beltrami operators.

Such convergence is called *totally normal*.

The second part of this paper derives several corollaries from this general convergence result, such as: uniform convergence of geodesics on compact sets, convergence of solutions to the Dirichlet problem (generalizing a result of Dziuk [7] who considered interpolating sequences of polyhedral meshes only), as well as weak convergence of polyhedral mean curvature. In particular, it is shown that a smooth limit surface of a sequence of discrete minimal surfaces is a minimal surface in the classical sense.

2 Approximating smooth surfaces

2.1 Polyhedral surfaces

A *polyhedral surface* $M_\tau \subset \mathbb{R}^3$ is a connected topological 2-manifold which is made up of flat triangles that are glued along their common edges such that no vertex appears in the interior of an edge. We only consider finite and connected triangulations. If $\gamma : [a, b] \to M_\tau$ is a continuous curve, then the *length* of γ is the supremum over all partitions $Z = \{t_0 = a \le t_1 \le ... \le t_n = b\}$ of $[a, b]$:

$$l(\gamma) = \sup_Z \sum_{i=1}^{n} d_{\mathbb{R}^3}(\gamma(t_{i-1}), \gamma(t_i)),$$

where $d_{\mathbb{R}^3}$ denotes the Euclidian metric of $\mathbb{R}^3$. Let x and y be two points in M_τ. Then the distance $d(x, y)$ between x and y is defined as

$$d(x, y) := \inf_\gamma l(\gamma), \tag{1}$$

the infimum taken over all continuous curves $\gamma : [a, b] \to M_\tau$. Following Gromov [10], M_τ equipped with this metric is called a *length space*. On individual triangles the length metric coincides with the induced flat metric from ambient $\mathbb{R}^3$. Across an edge of two adjacent triangles this metric is still flat as one can rotate those triangles about their common edge until they become coplanar. The situation changes at vertices where the metric exhibits *cone points*, cp. [24].

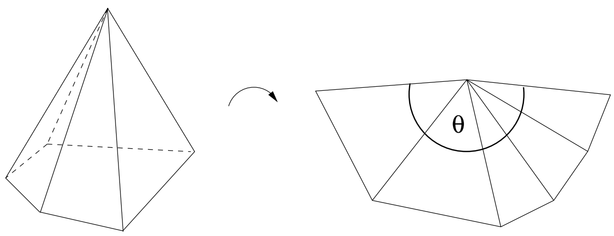

Figure 1: A neighborhood of a vertex with total vertex angle θ equipped with the length metric is isometric to a metric cone with cone angle θ.

Definition 1 (metric cone). The set $C_\theta := \{(r, \varphi)|0 \le r;\ \varphi \in \mathbb{R}/\theta\mathbb{Z}\}/_\sim$, where $(0, \varphi_1) \sim (0, \varphi_2)$, together with the (infinitesimal) metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$$

is called a *metric cone* with cone angle θ. The *cone point* is the coset consisting of all points $(0, \varphi) \in C_\theta$.

A cone point is called *singular* if the cone angle does not equal 2π. A singular cone point is *spherical* if the cone angle is less than 2π; otherwise it is *hyperbolic*. The cone metric $\mathrm{d}s^2$ is the infinitesimal version of the length metric, see Figure 1.

2.2 Normal graph and shortest distance map

In this paragraph the *shortest distance map* is introduced in order to compare a smooth surface to a polyhedral surface nearby.

Definition 2. Let $M \subset \mathbb{R}^3$ be a closed subset. The *medial axis* of M is the set of those points in $\mathbb{R}^3$ which do not have a unique closest neighbor in M. The *reach* of M is the distance of M to its medial axis.

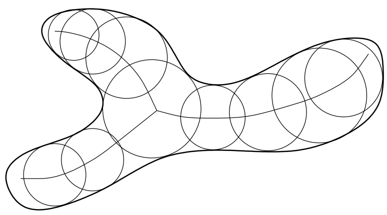

Figure 2: Medial axis of a smooth shape.

If $M \subset \mathbb{R}^3$ is a smoothly embedded surface then its medial axis corresponds to the locus of centers of spheres tangentially touching M in at least two points without intersecting M (see Figure 2); and the reach of M is the infimum over the radii of such spheres. The reach of a smooth surface M is bounded above by the radii of osculating spheres of M:

$$\operatorname{reach}(M) \leq \inf_{x \in M} \frac{1}{|\kappa|_{max}(x)}, \tag{2}$$

where $|\kappa|_{max}(x)$ denotes the maximal absolute value of the normal curvatures at $x \in M$. Note that a compact and smoothly embedded surface M always has positive reach (but a polyhedron does not). For a general treatment of sets of *positive reach* we refer to Federer [8].

The notion of reach serves as the core tool for parameterizing polyhedral surfaces directly over smooth manifolds.

Definition 3 (normal graph). A polyhedral surface M_τ is a *normal graph* over the smooth surface M if it is within the reach of M and the map which maps each point on M_τ to its closest point on M is a bijection, see Figure 3.

Then there is a bijection $\Phi : M \to M_\tau$ which takes $x \in M$ to the intersection point $\Phi(x) \in M_\tau$ of the normal line through x with the polyhedral surface M_τ. We call this map *shortest distance map*. The map Φ naturally splits into a tangential and a normal component:

$$\Phi(x) = Id_M(x) + \phi(x) \cdot N(x), \tag{3}$$

where N is the oriented normal of M, Id_M is the embedding of M into $\mathbb{R}^3$ and ϕ is the scalar-valued distance function.

2.3 The metric distortion tensor

The shortest distance map Φ induces a metric on M which allows to compare the spaces M_τ and M as metric spaces. Define a metric g_τ on M by

$$g_\tau(X, Y) := g_{M_\tau}(\,\mathrm{d}\Phi(X),\, \mathrm{d}\Phi(Y)) = \langle\, \mathrm{d}\Phi(X),\, \mathrm{d}\Phi(Y)\rangle_{\mathbb{R}^3} \quad \text{a.e.}, \tag{4}$$

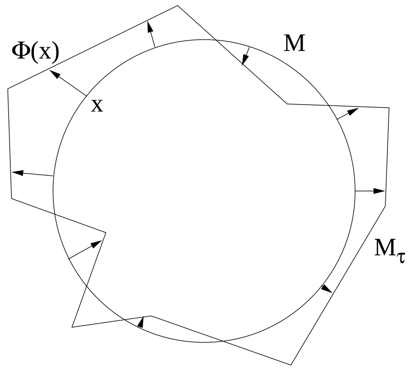

Figure 3: M_τ is a normal graph over M. At each point $x \in M$, the map Φ takes x to the intersection of the normal line through x with the polyhedral surface M_τ. The inverse Φ^{-1} thus realizes the shortest distance from M_τ to M.

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the standard inner product on $\mathbb{R}^3$.

Definition 4 (metric distortion tensor). There exists a symmetric positive definite 2×2 matrix field $A(x)$, $x \in M$, uniquely defined M-almost everywhere, such that for all vector fields X and Y on M

$$g_\tau(X, Y) = g(A(X), Y) \quad \text{a.e.} \tag{5}$$

The metric distortion tensor A is smooth on the pre-image of the interior of triangles of M_τ. The next theorem shows that A only depends on the distance between the surfaces M and M_τ, the angle between their normals and the curvature of the smooth surface M.

Theorem 1 (geometric splitting of metric distortion tensor). *Let M_τ be a closed polyhedral surface with normal field N_τ which is a normal graph over an embedded, closed, smooth surface M with normal field N. Then the metric distortion tensor A satisfies*

$$A = P \circ Q^{-1} \circ P \quad \textit{a.e.}, \tag{6}$$

a decomposition into symmetric positive definite matrices P and Q which can be diagonalized (possibly in different ON-frames) to take the form

$$P = \begin{pmatrix} 1 - \phi \cdot \kappa_1 & 0 \\ 0 & 1 - \phi \cdot \kappa_2 \end{pmatrix} \tag{7}$$

$$Q = \begin{pmatrix} \langle N, N_\tau \circ \Phi \rangle^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{8}$$

where κ_1 and κ_2 denote the principal curvatures of the smooth manifold M and ϕ is as in equation (3).

Remark 1. The matrix P is *positive definite* by the assumption that M_τ is in the reach of M since $1 - \phi \cdot \kappa_i > 0$ by inequality (2).

Theorem 1. It suffices to work over a single triangle T of M_τ. Consider the map $\Psi = \Phi^{-1} : M_\tau \to M$. For any map $f : M \to \mathbb{R}$, let $f_T : T \to \mathbb{R}$ denote the pullback $f_T = f \circ \Psi_{|T}$ to the triangle T. Then Ψ can be written as

$$\Psi = Id_{M_\tau} - \phi_T \cdot N_T. \tag{9}$$

Note that N_T stands for the pullback of N to the triangle T, rather than the normal N_τ to T. Differentiating (9) yields

$$\mathrm{d}\Psi = Id_{M_\tau} - N_T \cdot \mathrm{d}\phi_T - \phi_T \cdot \mathrm{d}N_T, \tag{10}$$

where the differential d is taken with respect to the canonical smooth structure on T. But

$$\mathrm{d}N_T = \mathrm{d}_M\, N \circ \mathrm{d}\Psi = -\mathbf{S} \circ \mathrm{d}\Psi, \tag{11}$$

where d_M denotes the outer differential on M, and $\mathbf{S} = -\,\mathrm{d}_M N$ is the Weingarten operator on M. Formulas (10) and (11) imply that

$$\mathrm{d}\Psi = (Id_M - \phi \cdot \mathbf{S})^{-1} \circ (Id_{M_\tau} - N_T \cdot \mathrm{d}\phi_T). \tag{12}$$

The tangent spaces $\mathrm{T}_x M$ and $\mathrm{T}_{\Phi(x)} T$ are linear subspaces of $\mathbb{R}^3$. Let

$$P = (Id_M - \phi \cdot \mathbf{S}) : \quad \mathrm{T}_x M \to \mathrm{T}_x M, \tag{13}$$

$$\tilde{Q} = (Id_{M_\tau} - N_T \cdot \mathrm{d}\phi_T) : \quad \mathrm{T}_{\Phi(x)} T \to \mathrm{T}_x M. \tag{14}$$

Then P is a *symmetric* endomorphism of $\mathrm{T}_x M$, and $\tilde{Q}$ is a linear map from $\mathrm{T}_{\Phi(x)} T$ to $\mathrm{T}_x M$. By equation (12),

$$\mathrm{d}\Phi = \mathrm{d}\Psi^{-1} = \tilde{Q}^{-1} \circ P. \tag{15}$$

For vectors $X_1, X_2 \in \mathrm{T}_x M$, define the *symmetric* endomorphism Q of $\mathrm{T}_x M$ by

$$\langle Q^{-1}(X_1), X_2 \rangle_{\mathbb{R}^3} = \langle \tilde{Q}^{-1}(X_1), \tilde{Q}^{-1}(X_2) \rangle_{\mathbb{R}^3}.$$

By the definition of the metric distortion tensor A and equation (15) it follows that

$$\begin{aligned} \langle A(X_1), X_2 \rangle_{\mathbb{R}^3} &= \langle\, \mathrm{d}\Phi(X_1),\, \mathrm{d}\Phi(X_2) \rangle_{\mathbb{R}^3} \\ &= \langle PQ^{-1}P(X_1), X_2 \rangle_{\mathbb{R}^3}, \end{aligned}$$

proving (6). Equation (7) follows from (13). It remains to show that for $X_1, X_2 \in \mathrm{T}_x M$ the quadratic form

$$\langle \tilde{Q}^{-1}(X_1), \tilde{Q}^{-1}(X_2) \rangle_{\mathbb{R}^3}$$

on $\mathrm{T}_x M$ can be diagonalized as stated in equation (8). To show that, let Y be a vector field on the triangle T. Applying (10) and taking into account that as subspaces of $\mathbb{R}^3$, $N_T \perp \mathrm{im}(\mathrm{d}\Psi)$ and $N_T \perp \mathrm{im}(\mathrm{d}N_T)$, it follows that

$$0 = \langle\, \mathrm{d}\Psi(Y), N_T \rangle = \langle Y, N_T \rangle -\, \mathrm{d}\phi_T(Y). \tag{16}$$

Equations (14) and (16) then imply that $\tilde{Q}$ is the projection operator

$$\tilde{Q}(Y) = Y - N_T \cdot \langle N_T, Y \rangle. \tag{17}$$

So, if α denotes the angle between the normal $N(x)$ to M and $N_\tau(\Phi(x))$ to M_τ, then the quadratic form $\langle \tilde{Q}(Y_1), \tilde{Q}(Y_2) \rangle_{\mathbb{R}^3}$ has eigenvalues 1 and $\cos\alpha = \langle N(x), N_\tau(\Phi(x)) \rangle$, which implies (8). $\square$

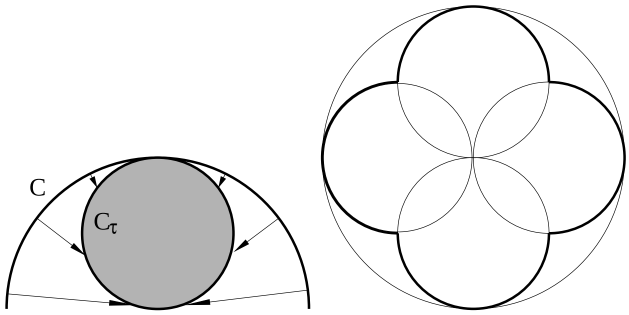

Figure 4: The shortest distance map may induce isometrics between non-congruent shapes. Left: The shortest distance map of the half unit circle C induces an isometry to a circle C_τ of radius 1/2. Right: By patching together pieces of the left picture one gets isometrics of the unit circle with a *dented circle.*

Corollary 1 (area distortion). *Under the assumptions of Theorem 1, the volume elements of M and M_τ satisfy*

$$\frac{\mathrm{d}vol_\tau}{\mathrm{d}vol} = (\det A)^{1/2} = \frac{1 + \phi^2 \cdot \kappa - \phi \cdot \mathrm{H}}{\langle N, N_\tau \circ \Phi \rangle} \quad a.e., \tag{18}$$

where κ denotes the Gauß curvature, and H *denotes the mean curvature of M.*

Proof. Equation (18) follows immediately from the explicit representation of the distortion tensor A in Theorem 1, and by using that $\kappa = \kappa_1 \cdot \kappa_2$ as well as $\mathrm{H} = \kappa_1 + \kappa_2$. □

By bounding the smallest and largest eigenvalues of A one also finds that

Corollary 2 (length distortion). *The infinitesimal distortion of length satisfies*

$$\min_i(1 - \phi \cdot \kappa_i) \leq \frac{\mathrm{d}l_\tau}{\mathrm{d}l} \leq \frac{max_i(1 - \phi \cdot \kappa_i)}{\langle N, N_\tau \circ \Phi \rangle} \quad a.e. \tag{19}$$

Remark 2 *(dented circle).* Note that even if the metric distortion induced by the shortest distance map equals the identity (so that the surfaces are isometric) the surfaces still need not be congruent. For example, consider the half unit circle $C = \{(\cos t, \sin t) : t \in [0, \pi]\}$. Any normal graph C_τ over C can then be written as

$$C_\tau = \left\{((1 - \phi(t)) \cdot \cos t, (1 - \phi(t)) \cdot \sin t) \in \mathbb{R}^2 : t \in [0, \pi]\right\},$$

where ϕ is the (signed) shortest distance map from C to C_τ along the unit circle's (outward) normal. Setting

$$\phi(t) := 1 - \sin t$$

one readily checks that C_τ becomes a circle of radius 1/2 and with center (0, 1/2), compare Figure 4. The inner product between the normals N of C and N_τ of C_τ is given by

$$\langle N, N_\tau \rangle = \frac{\langle N, \partial^2 C_\tau / \partial t^2 \rangle}{||\partial^2 C_\tau / \partial t^2||} = \sin t = 1 - \phi(t). \tag{20}$$

Let κ denote the curvature of C. The metric distortion between the two planar curves C and C_τ with respect to the shortest distance map Φ is then given by

$$a = \frac{1 - \phi \cdot \kappa}{\langle N, N_\tau \rangle} = \frac{1 - \phi}{\langle N, N_\tau \rangle} = 1, \tag{21}$$

where the last equality follows from (20). Note that formula (21) is a special case of (6). Hence, in this case, the metric distortion a between C and C_τ is the identity, so that the shortest distance map induces an isometry, although the shapes of C and C_τ are not congruent.

3 Convergence

Under the assumption of convergence in Hausdorff distance of a sequence of polyhedral surfaces to a smooth surface, we show that the following conditions are equivalent: (*i*) convergence of normals, (*ii*) convergence of the metric distortion tensors, (*iii*) convergence of area, and (*iv*) convergence of the Laplace-Beltrami operators. The proof is based on translating these geometric conditions into algebraic properties of the metric distortion tensor as defined in (5). The equivalence of such algebraic conditions, in turn, is then derived from the splitting $A = P \circ Q^{-1} \circ P$ into a product of symmetric operators P and Q as in Theorem 1. We first set up the relevant terminology.

Hausdorff distance. Let $M_1, M_2 \subset \mathbb{R}^3$ be non empty subsets. Then the *Hausdorff distance* between M_1 and M_2 is defined as

$$\mathrm{d}_H(M_1, M_2) = \inf \{\varepsilon > 0 \mid M_1 \subset U_\varepsilon(M_2) \,\text{and}\, M_2 \subset U_\varepsilon(M_1)\},$$

where $U_\varepsilon(M) = \left\{x \in \mathbb{R}^3 \mid \exists y \in M : \, d(x,y) < \varepsilon\right\}$.

Definition 5 (totally normal convergence). A sequence of polyhedra $M_{\tau,n}$ is said to converge *normally* to a smooth surface M if the sequence of normal fields converges in L^∞ under the shortest distance maps Φ_n, i.e. $\|N_{\tau,n} \circ \Phi_n - N\|_\infty \to 0$. Normal convergence is called *totally normal* if the Hausdorff distances $\mathrm{d}_H(M, M_{\tau,n})$ also go to zero.

Convergence of metric tensors. Each element $M_{\tau,n}$ in the approximating sequence induces a metric on the smooth reference surface M determined by the respective distortion tensor A_n. For almost every $x \in M$, $A_n(x)$ is an endomorphism of $\mathrm{T}_x M$. Let $\|A_n\|_\infty = \operatorname{ess\,sup}_{x \in M} \|A_n(x)\|_{op}$. Then *convergence of metric tensors* means $\lim_{n\to\infty} \|A_n - Id\|_\infty = 0$.

Laplace-Beltrami Operators. Let $H_0^1(M)$ denote the Sobolev space of weakly differentiable functions u on the smooth surface M which either vanish along the (non empty) boundary of M or for which $\bar{u} = \int u \, \mathrm{d}vol = 0$ (if M has no boundary). Let $H^{-1}(M)$ be the dual space, and let $\langle \cdot | \cdot \rangle$ denote the dual pairing between H^{-1} and H_0^1. We will consider the space $H_0^1(M)$ to be equipped with the norm

$$\|u\|^2_{H_0^1(M)} = \int_M g(\nabla u, \nabla u) \, \mathrm{d}vol.$$

The shortest distance map Φ allows to pull back the polyhedral metric on M_τ to the smooth reference space M. In particular, M comes with two Laplace-Beltrami operators $\Delta, \Delta_\tau : H_0^1(M) \to H^{-1}(M)$, given by

$$\langle \Delta u | v \rangle = - \int_M g(\nabla u, \nabla v) \, \mathrm{d}vol \tag{22}$$

$$\langle \Delta_\tau u | v \rangle = - \int_M g(A^{-1} \nabla u, \nabla v)(\det A)^{1/2} \, \mathrm{d}vol. \tag{23}$$

Convergence of these operators is to be understood in the operator norm of linear bounded maps between the spaces $H_0^1(M)$ and $H^{-1}(M)$.

Remark 3. The definition of the Laplace-Beltrami operator Δ_τ is based on the pullback of the polyhedral metric g_{M_τ} to the smooth surface M. On the other hand, Δ_τ is equal to the pullback of an *intrinsically* defined Laplace-Beltrami operator Δ_{M_τ} on the polyhedral surfaces M_τ; however, we will omit its construction here. The main technical difficulty is to rigorously define the Sobolev space $H_0^1(M_\tau)$ on a polyhedral surface. For a very general scheme for defining H_0^1 (based on a version of Rademacher's theorem which assures that weak differentiability is preserved under bi-Lipschitz maps), we refer to Cheeger [3] and Ziemer [25].

We now prove the main convergence result for polyhedral surfaces.

Theorem 2 (equivalent conditions for convergence). *Let $M \subset \mathbb{R}^3$ be a compact smooth surface, and let $\{M_{\tau,n}\}$ be a sequence of polyhedral surfaces which are normal graphs over M and which converge to M in Hausdorff distance. Then the following conditions are equivalent:*

- **i** *Convergence of normals:* $\|N_{\tau,n} \circ \Phi_n - N\|_\infty \to 0$.
- **ii** *Convergence of metric tensors:* $\|A_n - Id\|_\infty \to 0$.
- **iii** *Convergence of area:* $\| \mathrm{d}vol_{\tau,n} - \mathrm{d}vol\|_\infty \to 0$.
- **iv** *Convergence of Laplace-Beltrami operators:* $\|\Delta_{\tau,n} - \Delta\|_{op} \to 0$.

Remark 4. Remark 2 describes a case in which the metric tensors converge but the surfaces themselves do not. Hence the prerequisite in Theorem 2 that the Hausdorff distance between the surfaces must go to zero cannot be dropped in general.

Theorem 2. For simplicity, we constrain ourselves to surfaces without boundary. The proof of the theorem is based on first translating the geometric conditions (ii), (iii) and (iv) into algebraic properties of the metric distortion tensors A_n: convergence of the metric tensors by definition means $\|A_n - Id\|_\infty \to 0$, convergence of area measure is equivalent to $\| \det A_n\|_\infty \to 1$, and Lemma 1 provides algebraic conditions for convergence of Laplace-Beltrami operators. In a second step one shows that these algebraic conditions are equivalent to convergence of normals. Let $A_n = P_n \circ Q_n^{-1} \circ P_n$ as in Theorem 1, and let $\bar{A}_n = (\det A_n)^{1/2} A_n^{-1}$. We claim that

$$\begin{aligned} \|A_n - Id\|_\infty \to 0 \iff \| \det A_n\|_\infty \to 1 \iff \|\bar{A}_n - Id\|_\infty \to 0 \\ \iff \|\mathrm{tr}(\bar{A}_n - Id)\|_\infty \to 0 \end{aligned}$$

are all equivalent conditions to normal convergence. By assumption the surfaces converge in Hausdorff distance, so that $\|P_n - Id\|_\infty \to 0$. Then from the diagonalization

$$Q_n = \begin{pmatrix} \langle N, N_{\tau,n} \circ \Phi_n \rangle^2 & 0 \\ 0 & 1 \end{pmatrix},$$

it becomes evident that all of the above algebraic expressions converge if and only if $\langle N, N_{\tau,n} \circ \Phi_n \rangle \to 1$ in L^∞ - which is normal convergence. To complete the proof of the theorem, it remains to give algebraic conditions for convergence of the Laplace-Beltrami operators. The next lemma provides an upper and a lower bound for the difference between Δ and Δ_τ in terms of the distortion tensor A. □

Lemma 1 (convergence of Laplace-Beltrami operators). *Let $M_\tau \subset \mathbb{R}^3$ be an embedded compact polyhedral surface which is a normal graph over a smooth embedded closed surface M with corresponding distortion tensor A, and let $\bar{A} := (\det A)^{1/2} A^{-1}$. Then*

$$\frac{1}{2}\|\mathrm{tr}(\bar{A} - Id)\|_\infty \le \|\Delta_\tau - \Delta\|_{op} \le \|\bar{A} - Id\|_\infty. \tag{24}$$

Proof. Let $\langle \cdot | \cdot \rangle$ denote the dual pairing between H^{-1} and H_0^1. The upper bound in (24) is a straightforward application of Hölder's inequality. By definitions (22) and (23), we have

$$\begin{aligned} |\langle (\Delta_\tau - \Delta)u | v \rangle| &= \left| \int_M g((\bar{A} - Id)\nabla u, \nabla v)\, \mathrm{d}vol \right| \\ &\le \|\bar{A} - Id\|_\infty \cdot \|u\|_{H_0^1} \|v\|_{H_0^1}. \end{aligned}$$

The proof of the lower bound in (24) is more technical. Define the compactum $K \subset M$ to be the pre-image under Φ of the 1-complex of M_τ (its edges and vertices). Then K is a measure zero set. For an arbitrary but fixed $x \in M \setminus K$ we will construct a family of functions (f_ε) in $H_0^1(M)$ such that

$$\lim_{\varepsilon \to 0} \frac{|\langle (\Delta_\tau - \Delta) f_\varepsilon | f_\varepsilon \rangle|}{\|f_\varepsilon\|_{H_0^1}^2} = \frac{1}{2}\mathrm{tr}(\bar{A} - Id)(x), \tag{25}$$

proving our claim since it implies $\|\Delta_\tau - \Delta\|_{op} \ge \frac{1}{2}\sup_{x \in M \setminus K} \mathrm{tr}(\bar{A} - Id)(x)$.

Let $D_\varepsilon(x) \subset M \setminus K$ be a small ε-ball around x, and define $F_\varepsilon \in H^1(M)$ in polar coordinates (r, φ) by

$$F_\varepsilon(r, \varphi) = \begin{cases} \varepsilon - r & \text{for} \quad r < \varepsilon \\ 0 & \text{else,} \end{cases}$$

compare Figure 5. Project F_ε to $f_\varepsilon \in H_0^1(M)$ by

$$f_\varepsilon = F_\varepsilon - \frac{1}{|M|} \int_M F_\varepsilon \, \mathrm{d}vol.$$

By construction, $g(\nabla f_\varepsilon, \nabla f_\varepsilon) = 1$ on $D_\varepsilon(x) \setminus \{x\}$ and $\nabla f_\varepsilon = 0$ on $M \setminus D_\varepsilon(x)$, so that

$$\|f_\varepsilon\|_{H_0^1}^2 = \int_M g(\nabla f_\varepsilon, \nabla f_\varepsilon)\, \mathrm{d}vol = |D_\varepsilon(x)|.$$

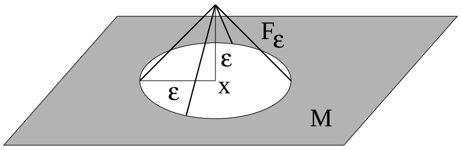

Figure 5: The family of functions (F_ε) for $\varepsilon \to 0$ gives a lower bound for the operator norm of the difference between the Laplacians.

On the other hand,

$$\langle (\Delta_\tau - \Delta) f_\varepsilon | f_\varepsilon \rangle = -\int_M g((\bar{A} - Id)\nabla f_\varepsilon, \nabla f_\varepsilon) \, \mathrm{d}vol,$$

so that equation (25) is equivalent to

$$\lim_{\varepsilon \to 0} \frac{1}{|D_\varepsilon(x)|} \int_{D_\varepsilon(x)} g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon) \, \mathrm{d}vol = \frac{1}{2} \mathrm{tr}(\bar{A})(x). \tag{26}$$

The idea is now to work first in the tangent space $\mathrm{T}_x M$ over x, and then use the exponential map to treat the general case. Let g_x denote the restriction of the metric tensor g to $\mathrm{T}_x M$. Let $\mathrm{d}vol_x$ denote the volume form on $\mathrm{T}_x M$, and let ∂_r denote the unit radial vector field on $\mathrm{T}_x M$. The matrix $\bar{A}_x := \bar{A}(x)$ acts as a linear map from $\mathrm{T}_x M$ to itself with eigenvalues λ and $1/\lambda$. Then on the ball of radius ε, $B_\varepsilon(0) \subset \mathrm{T}_x M$, we have

$$\begin{aligned} \int_{B_\varepsilon(0)} g_x(\bar{A}_x \partial_r, \partial_r) \, \mathrm{d}vol_x &= \int_0^\varepsilon \int_0^{2\pi} (\lambda \cos^2 \varphi + \frac{1}{\lambda} \sin^2 \varphi) r \, \mathrm{d}r \, \mathrm{d}\varphi \\ &= \frac{1}{2}(\lambda + \frac{1}{\lambda}) \cdot |B_\varepsilon(0)| \\ &= \frac{1}{2} \mathrm{tr} \bar{A}_x \cdot |B_\varepsilon(0)|, \end{aligned}$$

which proves (26) in the flat case (on $\mathrm{T}_x M$). To prove (26) in the general case, we define a 2-form ω_0 on $B_\varepsilon(0) \subset \mathrm{T}_x(M)$ and a 2-form ω_1 on $D_\varepsilon(x)$ by

$$\begin{aligned} \omega_0 &:= g_x(\bar{A}_x \partial_r, \partial_r) \, \mathrm{d}vol_x \\ \omega_1 &:= g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon) \, \mathrm{d}vol. \end{aligned}$$

To complete the proof, we are going to show that

$$\lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \omega_0 = \lim_{\varepsilon \to 0} \frac{1}{|D_\varepsilon(x)|} \int_{D_\varepsilon(x)} \omega_1. \tag{27}$$

Indeed, consider the exponential map $\exp_x : B_\varepsilon(0) \subset \mathrm{T}_x M \to D_\varepsilon(x)$. Let ω_1^* denote the pullback of ω_1 from $D_\varepsilon(x)$ to $B_\varepsilon(0)$, and let $\bar{A}^*$ be the pullback of $\bar{A}$ to $B_\varepsilon(0)$ under this map. By the Gauß Lemma, the exponential map is a radial isometry, so that $\mathrm{d}\exp(\partial_r) = \nabla f_\varepsilon$ and hence

$$\begin{aligned} g_x(\bar{A}^* \partial_r, \partial_r) &= g_{\exp(x)}(\bar{A}_{\exp(x)}(\mathrm{d}\exp(\partial_r)), \mathrm{d}\exp(\partial_r)) \\ &= g_{\exp(x)}(\bar{A}_{\exp(x)} \nabla f_\varepsilon, \nabla f_\varepsilon), \end{aligned}$$

Let $\mathrm{d}vol^*$ denote the pullback of the volume form $\mathrm{d}vol$ on $D_\varepsilon(x)$ to $B_\varepsilon(0)$. Then the last equation shows that

$$\omega_1^* = g_x(\bar{A}^*\partial_r, \partial_r)\,\mathrm{d}vol^*.$$

This implies $\|\omega_1^* - \omega_0\|_{\infty,B_\varepsilon(0)} \to 0$, so that

$$\begin{aligned}
&\left|\frac{1}{|D_\varepsilon(x)|}\int_{D_\varepsilon(x)} \omega_1 - \frac{1}{|B_\varepsilon(0)|}\int_{B_\varepsilon(0)} \omega_0\right| \\
&= \left|\frac{1}{|D_\varepsilon(x)|}\int_{B_\varepsilon(0)} \omega_1^* - \frac{1}{|B_\varepsilon(0)|}\int_{B_\varepsilon(0)} \omega_0\right| \\
&\leq \left|\frac{1}{|D_\varepsilon(x)|} - \frac{1}{|B_\varepsilon(0))|}\right| \int_{B_\varepsilon(0)} |\omega_1^*| + \frac{1}{|B_\varepsilon(0)|}\int_{B_\varepsilon(0)} |\omega_1^* - \omega_0| \\
&\leq \left|\frac{|B_\varepsilon(0)|}{|D_\varepsilon(x)|} - 1\right| \|\omega_1^*\|_{\infty,B_\varepsilon(0)} + \|\omega_1^* - \omega_0\|_{\infty,B_\varepsilon(0)} \longrightarrow 0,
\end{aligned}$$

proving (27) and hence our claim. □

4 Applications of normal convergence

In this section the general convergence results of Theorem 2 are applied to show convergence of discrete notions of geodesics, solutions to the Dirichlet problem as well as mean curvature.

4.1 Convergence of geodesics

Definition 6 (shortest geodesic). A shortest geodesic in a metric space (V, d) is a continuous curve $\gamma : [a, b] \to V$ such that $d(\gamma(t), \gamma(t')) = |t' - t|$ for all t and t' in the interval $[a, b]$.

The Hopf-Rinow theorem for metrically complete length spaces [10] asserts that any two points can be connected by a shortest geodesic. This ensures that the infimum over all curves which was used in (1) to define the distance between two points is actually attained as a minimum.

Corollary 3 (convergence of geodesics). *Let $\{M_{\tau,n}\}$ be a family of polyhedral surfaces converging totally normally to a compact smooth surface M with associated shortest distance maps Φ_n. Let $x, y \in M$ be two points, and let γ_n be a shortest geodesic connecting $\Phi_n(x)$ to $\Phi_n(y)$ on $M_{\tau,n}$. Then each accumulation point of $\{\gamma_n\}$ in the compact-open topology on $C^0(\mathbb{R}, \mathbb{R}^3)$ is a shortest geodesic on M. The set of such accumulation points is not empty. In particular, there exists a shortest geodesic γ on M and a sub-sequence of shortest geodesics (γ_{n_i}) on M_{τ,n_i} such that $\gamma_{n_i} \to \gamma$ uniformly.*

Proof. We consider all objects to be defined on the smooth reference surface M by using the pull-backs with the maps Φ_n. Let A_n denote the metric distortion tensor corresponding to $g_{\tau,n}$, and let $\underline{c}_n := 1/\|A_n^{-1}\|_\infty^{1/2}$ and $\overline{c}_n := \|A_n\|_\infty^{1/2}$. If β is a Lipschitz curve on M, then the $g_{\tau,n}$-length $l_n(\beta)$ and the g-length $l(\beta)$ are related by

$$\underline{c}_n \cdot l(\beta) \leq l_n(\beta) \leq \overline{c}_n \cdot l(\beta).$$

The distance between the points x and y equals the infimum over the length of all Lipschitz curves connecting these points. The last inequality then implies:

$$\underline{c}_n \cdot d(x,y) \leq d_n(x,y) \leq \overline{c}_n \cdot d(x,y).$$

Hence, if γ_n is a shortest geodesic connecting x and y in the $g_{\tau,n}$-metric, then

$$\begin{aligned} \underline{c}_n \cdot d(x,y) &\leq d_n(x,y) = l_n(\gamma_n) \leq \overline{c}_n \cdot l(\gamma_n) \\ \overline{c}_n \cdot d(x,y) &\geq d_n(x,y) = l_n(\gamma_n) \geq \underline{c}_n \cdot l(\gamma_n). \end{aligned}$$

This implies

$$\frac{\underline{c}_n}{\overline{c}_n} \cdot d(x,y) \leq l(\gamma_n) \leq \frac{\overline{c}_n}{\underline{c}_n} d(x,y).$$

By assumption, $\underline{c}_n \to 1$ and $\overline{c}_n \to 1$, so that

$$l(\gamma_n) \to d(x,y). \tag{28}$$

Now, assume γ is an accumulation point of $\{\gamma_n\}$. Since the length functional $l : C^0(\mathbb{R}, \mathbb{R}^3) \to \mathbb{R}$ is lower semi-continuous, (28) implies

$$l(\gamma) \leq \liminf l(\gamma_n) = d(x,y).$$

Hence γ is indeed a shortest geodesic connecting x to y.
It remains to show that the set of such accumulation points is not empty. But

$$d(\gamma_n(t), \gamma_n(t')) \leq \frac{1}{\underline{c}_n} \cdot d_n(\gamma_n(t), \gamma_n(t')) = \frac{1}{\underline{c}_n} \cdot |t - t'|,$$

for each t, t' in the domain of γ_n. Hence the family $\{\gamma_n\}$ is equicontinuous. Since $|t-t'|$ can be bounded by $\sup_n \operatorname{diam}(M_{\tau,n}) \leq \sup_n \overline{c}_n \cdot \operatorname{diam}(M)$, it follows from the Arzela-Ascoli theorem that there is an accumulation point in the compact-open topology on $C^0(\mathbb{R}, \mathbb{R}^3)$. □

4.2 Convergence of the Dirichlet problem

Assume M_τ is a normal graph over M and that both surfaces have non empty boundary. Given $f \in L^2(M)$, the *Dirichlet problem* with respect to the elliptic operators Δ, $\Delta_\tau : H_0^1(M) \to H^{-1}(M)$ is to find $u, u_\tau \in H_0^1(M)$ such that

$$\langle \Delta u | \varphi \rangle = \int_M f \varphi \, \mathrm{d}vol \quad \forall \varphi \in C_0^\infty(M) \tag{29}$$

$$\langle \Delta_\tau u_\tau | \varphi \rangle = \int_M f \varphi \, \mathrm{d}vol_\tau \quad \forall \varphi \in C_0^\infty(M), \tag{30}$$

where $\mathrm{d}vol_\tau = (\det A)^{1/2} \, \mathrm{d}vol$.

Remark 5. For compact surfaces without boundary the right hand sides have to be adjusted by replacing f by $(f - \overline{f})$ in (29), respectively $(f - \overline{f}^\tau)$ in (30), where $\overline{f} := \frac{1}{|M|} \int_M f \, \mathrm{d}vol$ and $\overline{f}^\tau := \frac{1}{|M_\tau|} \int_M f \, \mathrm{d}vol_\tau$. In what follows we will treat surfaces with non empty boundary explicitly and remark on the necessary adjustments for surfaces without boundary.

Remark 6. We will make repeated use of the following identity: Let A be a positive definite, symmetric 2×2 matrix. Then $(\det A)^{1/2}A^{-1}$ has positive eigenvalues λ and $1/\lambda$ and $\|(\det A)^{1/2}A^{-1} - Id\|_{op} = \|(\det A)^{1/2}A^{-1}\|_{op} - 1$.

Corollary 4 (explicit consistency error). *Let the compact polyhedral surface M_τ be a normal graph over the smooth surface M, and let A be the distortion tensor. Assume M and M_τ have non empty boundary. For $f \in L^2(M)$, let u, u_τ be solutions to the Dirichlet problems (29) and (30). Then*

$$\|u - u_\tau\|_{H_0^1} \leq \left((C_A - 1) + \left\|1 - (\det A)^{1/2}\right\|_\infty\right) \cdot C_A \cdot \|E\|_{op} \cdot \|f\|_{L^2},$$

where $C_A := \|(\det A)^{1/2}A^{-1}\|_\infty$ and $E : H_0^1(M) \hookrightarrow L^2(M)$ denotes the natural embedding.

Proof. Define two operators $E^*, E_\tau^* : L^2(M) \to H^{-1}(M)$ by

$$\langle E^*(f)|\varphi\rangle = \int_M f\varphi \,\mathrm{d}vol$$
$$\langle E_\tau^*(f)|\varphi\rangle = \int_M f\varphi \,\mathrm{d}vol_\tau.$$

Then the Dirichlet problems amount to solving

$$\Delta u = E^*(f) \quad \text{and} \quad \Delta_\tau u_\tau = E_\tau^*(f). \tag{31}$$

As linear maps from $H_0^1(M)$ to $H^{-1}(M)$, the operators Δ and Δ_τ are elliptic and bounded; the ellipticity constant of Δ equals 1 and the ellipticity constant of Δ_τ is $1/C_A$. By the Lax-Milgram Lemma, both operators can be inverted. Hence,

$$\begin{aligned}\|u - u_\tau\|_{H_0^1} &= \|\Delta^{-1}E^*(f) - \Delta_\tau^{-1}E_\tau^*(f)\|_{H_0^1} \\ &= \|\Delta^{-1}E^*(f) - \Delta_\tau^{-1}E^*(f) + \Delta_\tau^{-1}E^*(f) - \Delta_\tau^{-1}E_\tau^*(f)\|_{H_0^1} \\ &\leq \|\Delta^{-1} - \Delta_\tau^{-1}\|_{op}\|E^*(f)\|_{H^{-1}} + \|\Delta_\tau^{-1}\|_{op}\|E^*(f) - E_\tau^*(f)\|_{H^{-1}}.\end{aligned}$$

We now examine these terms one by one. First,

$$\|\Delta^{-1} - \Delta_\tau^{-1}\|_{op} \leq \|\Delta^{-1}\|_{op} \cdot \|\Delta - \Delta_\tau\|_{op} \cdot \|\Delta_\tau^{-1}\|_{op}.$$

From Lemma 1 and Remark 6 we know that $\|\Delta - \Delta_\tau\|_{op} \leq (C_A - 1)$. As the ellipticity constant of $-\Delta_\tau$ is $1/C_A$ it follows that

$$\|\Delta_\tau^{-1}\|_{op} \leq C_A. \tag{32}$$

Hence,

$$\|\Delta^{-1} - \Delta_\tau^{-1}\|_{op} \leq C_A \cdot (C_A - 1). \tag{33}$$

Secondly, let $\|E\|_{op}$ denote the embedding constant of $E : H_0^1(M) \hookrightarrow L^2(M)$. E^* is the adjoint operator to E, so that

$$\|E^*(f)\|_{H^{-1}} \leq \|E\|_{op} \cdot \|f\|_{L^2}. \tag{34}$$

Finally, from the definitions of E^* and E_τ^* it follows that

$$\|E^*(f) - E_\tau^*(f)\|_{H^{-1}} \leq \|1 - (\det A)^{1/2}\|_\infty \cdot \|E\|_{op} \cdot \|f\|_{L^2}. \tag{35}$$

Combining (32), (33), (34), and (35) proves the estimate stated in the corollary. □

Corollary 5 (convergence of Dirichlet problem). *If a sequence of polyhedral surfaces* $\{M_{\tau,n}\}$ *converges totally normally to* M*, then the solutions to the respective Dirichlet problems converge in* H^1_0.

Remark 7. Corollary 4 generalizes a result of Dziuk [7] who proves h^2-estimates for *interpolating* sequences of polyhedral surfaces (interpolating meaning that the vertices of the approximating sequence of meshes all reside on the smooth surface M).

Remark 8. Using the operator form (31) of the Dirichlet problem instead of the classical definition has the advantage that the above convergence proof easily carries over to surfaces without boundary. In fact, only the definitions of E^* and E^*_τ and the estimate of $\|E^*(f) - E^*_\tau(f)\|_{H^{-1}}$ have to be slightly adjusted in what was said above.

4.2.1 Convergence of Galerkin scheme

For numerical purposes, the solution u_τ is approximated by a finite element solution u_h. We quickly review how to compute u_h explicitly, and then show that the total error $\|u - u_h\|$ must go to zero under totally normal convergence. Again, we only treat the case $\partial M \neq \emptyset$ explicitly. As in the planar case, a *Galerkin scheme* on a polyhedral surface M_τ is defined by restricting the space of test functions as well as the space of solutions of the Dirichlet problem to the same *finite-dimensional* subspace $S_{h,0} \subset H^1_0(M_\tau)$.

Definition 7 (finite element space). For vertices $p \in M_\tau \setminus \partial M_\tau$ and $q \in M_\tau$ define

$$\phi_p(q) := \begin{cases} 1 & \text{for} \quad q = p \\ 0 & \text{for} \quad q \neq p, \end{cases}$$

and extended ϕ_p to all of M_τ by linear interpolation on triangles. The finite element space $S_{h,0} \subset H^1_0(M_\tau)$ is spanned by the functions $\{\phi_p\}$.

Every $u_h \in S_{h,0}$ can be written as $u_h = \sum_q u^q_h \phi_q$ with coefficients u^q_h. Let

$$\Delta_{pq} := -\int_{M_\tau} g_\tau(\nabla_\tau \phi_p, \nabla_\tau \phi_q)\, \mathrm{d}vol_\tau \quad \text{and} \quad \mathrm{b}_p := \int_{M_\tau} f \cdot \phi_p \, \mathrm{d}vol_\tau.$$

Then the Dirichlet problem becomes a *finite linear problem*: Find the vector (u^q_h) satisfying

$$\sum_q \Delta_{pq} u^q_h = \mathrm{b}_p. \tag{36}$$

One readily verifies the *cotan* representation (cp. [16]) of Δ_{pq}.

Lemma 2 (cotan formula). *The non-zero entries of the discrete Laplacian on a polyhedral surface* $M_\tau \subset \mathbb{R}^3$ *are given by*

$$\Delta_{pq} = \frac{1}{2}(\cot \alpha_{pq} + \cot \beta_{pq}) \quad \textit{and} \quad \Delta_{pp} = -\sum_{q_i \in link(p)} \Delta_{pq_i}, \tag{37}$$

if p *and* q *share an edge, and where* α_{pq} *and* β_{pq} *denote the vertex angles opposite to the edge* (pq) *in the two triangles adjacent to* (pq).

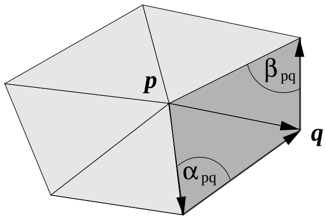

Figure 6: Only the angles α_{pq} and β_{pq} enter into the expression for Δ_{pq}.

Since the surface M_τ consists of flat triangles, the error $\|u_h - u_\tau\|_{H_0^1}$ can be treated exactly as in the planar case [5] (it only depends on the aspect ratio of the triangles comprising M_τ),

$$\|u_\tau - u_h\|_{L^2} + h\|u_\tau - u_h\|_{H_0^1} \leq C \cdot h^2 \cdot \|f\|_{L^2}.$$

Then, together with the explicit estimate of Corollary 4, the *total error* of the Galerkin scheme simply follows from the triangle inequality

$$\|u - u_h\|_{H_0^1} \leq \|u - u_\tau\|_{H_0^1} + \|u_\tau - u_h\|_{H_0^1}. \tag{38}$$

Corollary 6 (convergence of cotan formula). *If a sequence of polyhedral surfaces $\{M_{\tau,n}\}$ totally normally converges to M, then the solutions to the finite element Dirichlet problems (36) converge in norm to the solution of (29).*

4.3 Convergence of Mean Curvature

We discuss convergence of *weak* and *discrete* mean curvature. Weak mean curvature corresponds to mean curvature viewed as a *functional*. Discrete mean curvature will denote the corresponding *PL-function*. In this section we show that the functional does converge, whereas the corresponding function does in general not converge. We treat the two cases $\partial M = \emptyset$ and $\partial M \neq \emptyset$ simultaneously.

Definition 8 (weak mean curvature). Let $\vec{I} : M \to \mathbb{R}^3$ be the embedding of M, and let $\vec{I}_\tau := \Phi \circ \vec{I} : M \to \mathbb{R}^3$ denote the embedding of M_τ. Then the weak mean curvatures are functionals defined by

$$\vec{H} := \Delta \vec{I} \in (H^{-1}(M))^3,$$
$$\vec{H}_\tau := \Delta_\tau \vec{I}_\tau \in (H^{-1}(M))^3.$$

Weak mean curvature is a 3-*valued* functional, with norm defined as

$$\|\vec{H}\|_{H^{-1}} := \sup_{0 \neq u \in H_0^1} \frac{\|\langle \vec{H} | u \rangle\|_{\mathbb{R}^3}}{\|u\|_{H_0^1}}.$$

Corollary 7 (convergence of weak mean curvature). *Let M_τ be normal graph over the closed smooth surface M with distortion tensor A and shortest distance map Φ. Then*

$$\|\vec{H} - \vec{H}_\tau\|_{H^{-1}} \leq \sqrt{|M|} \cdot (C_A - 1 + C_A\|Id - \mathrm{d}\Phi\|_\infty), \tag{39}$$

where $C_A = \|(\det A)^{1/2} A^{-1}\|_\infty$, $|M|$ is the total area of M, and $\|Id - \mathrm{d}\Phi\|_\infty$ denotes the essential supremum over the pointwise operator norm of the operator $(Id - \mathrm{d}\Phi)(x) : \mathrm{T}_x M \to \mathbb{R}^3$. Hence if a sequence of polyhedral surfaces converges to M totally normally then their weak mean curvatures converge in H^{-1}.

Proof. Consider the triangle inequality

$$(\Delta\vec{I} - \Delta_\tau\vec{I}_\tau) = (\Delta\vec{I} - \Delta_\tau\vec{I}) + (\Delta_\tau\vec{I} - \Delta_\tau\vec{I}_\tau).$$

The second term can be estimated by Hölder's inequality

$$\begin{aligned}\left\|\langle\Delta\vec{I} - \Delta_\tau\vec{I}|u\rangle\right\|_{\mathbb{R}^3} &= \left\|\int_M (A^{-1}(\det A)^{1/2} - Id)\nabla u \,\mathrm{d}vol\right\|_{\mathbb{R}^3} \\ &\le \sqrt{|M|}\cdot(C_A - 1)\cdot\|u\|_{H_0^1}.\end{aligned}$$

For the last term, note that $\langle\nabla\vec{I}, \nabla u\rangle = \nabla u$ and $\langle\nabla\vec{I}_\tau, \nabla u\rangle = \mathrm{d}\Phi(\nabla u)$. Hence

$$\begin{aligned}\left\|\langle\Delta_\tau\vec{I} - \Delta_\tau\vec{I}_\tau|u\rangle\right\|_{\mathbb{R}^3} &= \left\|\int_M \left\langle(\nabla\vec{I} - \nabla\vec{I}_\tau), A^{-1}(\det A)^{1/2}\nabla u\right\rangle \mathrm{d}vol\right\|_{\mathbb{R}^3} \\ &\le \sqrt{|M|}\cdot C_A\cdot\|Id - \mathrm{d}\Phi\|_\infty\cdot\|u\|_{H_0^1},\end{aligned}$$

proving (39). For the convergence statement, it remains to show that totally normal convergence implies $\|Id - \mathrm{d}\Phi\|_\infty \to 0$. To show that, we work over a single triangle T of M_τ. Let $N_T = N \circ \Phi^{-1}$ denote the pullback of the normal field N on M to the triangle T. Then from equation (15) we know that $\mathrm{d}\Phi = \tilde{Q}^{-1} \circ P$, where P is as in Theorem 1 and $\tilde{Q}$ is given by $\tilde{Q}(Y) = Y - N_T \cdot \langle N_T, Y\rangle$, cp. equation (17). Then totally normal convergence implies $P \to Id$ and $\tilde{Q} \to Id$, and hence $\mathrm{d}\Phi \to Id$. ☐

4.3.1 Polyhedral minimal surfaces

In [16], Pinkall and Polthier for the first time started a systematic treatment of discrete minimal surfaces. Their approach has spawned a rich pool of examples of discrete minimal surfaces, cp. [12] [11] [19]. In this section we show that if sequences of discrete minimal surfaces converge to a smooth surfaces in Hausdorff distance, then the smooth limit surface must be a minimal surface in the classical sense.

We first note that weak mean curvature can be computed explicitly on a polyhedral surface by evaluation on the nodal basis (compare Figure 6 for the notation of angles):

$$\langle\vec{H}_\tau|\phi_p\rangle = \frac{1}{2}\sum_{q\in link(p)} (q - p)(\cot\alpha_{pq} + \cot\beta_{pq}) \quad \forall p \in M_\tau \setminus \partial M_\tau. \tag{40}$$

A polyhedral surface is called *minimal* if $\langle\vec{H}_\tau|\phi_p\rangle = 0$ for all $p \in M_\tau \setminus \partial M_\tau$. If all degrees of freedom are put into the vertices, then polyhedral minimal surfaces are - just like their smooth counterparts - critical points of the area functional.

Although discrete minimality only requires that $\langle\vec{H}_\tau|u_h\rangle = 0$ for all u_h in the finite element space S_h, so that it is a *weaker* condition than $\vec{H}_\tau = 0$, we have the following convergence result:

Corollary 8 (convergence of polyhedral minimal surfaces). *Let $\{M_{\tau,n}\}$ be a sequence of polyhedral minimal surfaces converging totally normally to a smooth closed surface $M \subset \mathbb{R}^3$. Assume the aspect ratios of all triangles in this sequence are uniformly bounded above. Then M is a minimal surface in the classical sense.*

Proof. Let $\vec{H}$ denote the mean curvature of the smooth surface M. Let $u \in H_0^1(M)$ and let $u_n \in S_{h_n,0}$ be the projection of u to the finite-element space $S_{h_n,0} \subset H_0^1(M)$ induced by $M_{\tau,n}$. Then

$$\left\|\langle \vec{H}|u\rangle\right\|_{\mathbb{R}^3} \leq \left\|\langle \vec{H}|u-u_n\rangle\right\|_{\mathbb{R}^3} + \left\|\langle \vec{H}|u_n\rangle\right\|_{\mathbb{R}^3}. \tag{41}$$

We are going to show that $\langle \vec{H}|u\rangle = 0$ by showing that the right hand side of (41) must vanish. Since $\vec{H}$ is smooth,

$$\begin{aligned}\left\|\langle \vec{H}|u-u_n\rangle\right\|_{\mathbb{R}^3} &= \left\|\int_M \vec{H}\cdot(u-u_n)\,\mathrm{d}vol\right\|_{\mathbb{R}^3} \\ &\leq \left\|\vec{H}\right\|_{L^2(M)} \|u-u_n\|_{L^2(M)}.\end{aligned}$$

But $\|u-u_n\|_{L^2(M)} \leq C\cdot h_n \cdot \|u\|_{H_0^1(M)} \to 0$, where h_n denotes the longest edge length of the triangulation of $M_{\tau,n}$, and C is independent of n because the sequence is assumed to have bounded aspect ratio (one argues just like in the planar case [5]). To estimate the last term in (41), let $\vec{H}_{\tau,n}$ denote the weak mean curvature associated with $M_{\tau,n}$. By assumption $\vec{H}_{\tau,n}$ vanishes on $S_{h_n,0}$ and hence

$$\begin{aligned}\left\|\langle \vec{H}|u_n\rangle\right\|_{\mathbb{R}^3} &= \left\|\langle \vec{H}-\vec{H}_{\tau,n}|u_n\rangle\right\|_{\mathbb{R}^3} \\ &\leq \left\|\vec{H}-\vec{H}_{\tau,n}\right\|_{H^{-1}} \cdot \|u_n\|_{H_0^1(M)}.\end{aligned}$$

From Corollary 7 it follows that $\|\vec{H}-\vec{H}_{\tau,n}\|_{H^{-1}} \to 0$. Since u_n is a projection of u, it follows that $\|u_n\|_{H_0^1} \leq \|u\|_{H_0^1}$, so that $\|\langle \vec{H}|u_n\rangle\|_{\mathbb{R}^3} \to 0$. From (41) we then get $\vec{H} = 0$, as asserted. □

4.3.2 Discrete Mean Curvature

Weak mean curvature is a 3-valued functional. *Discrete mean curvature* is the 3-valued *PL-function* associated with this functional. Corollary 7 shows that the mean curvature functionals converges in H^{-1}. However, the objective of this section is to show that the discrete mean curvature functions in general fail to converge in L^2.

Definition 9 (discrete mean curvature). Discrete mean curvature is the 3-valued PL-function $\vec{H}_{dis} \in S_{h,0}$ defined by

$$\int_{M_\tau} \vec{H}_{dis}\cdot u_h\,\mathrm{d}vol_\tau = \langle \vec{H}_\tau|u_h\rangle \quad \forall\, u_h \in S_{h,0}. \tag{42}$$

Note, only because the dimension of $S_{h,0}$ is finite, it is possible to associate a discrete function to the mean curvature functional. In general, there is no infinite-dimensional analogue of this construction. The discrete mean curvature function can be computed *explicitly* on a polyhedral surface M_τ:

$$\vec{H}_{dis} = \sum_{p,q \in M_\tau} \langle \vec{H}_\tau | \phi_p \rangle M^{pq} \phi_q, \tag{43}$$

where $\langle \vec{H}_\tau | \phi_p \rangle$ denotes the evaluation of the mean curvature functional $\vec{H}_\tau$ on the nodal basis function ϕ_p as in (40), and M^{pq} denotes the inverse of the *mass matrix* M_{pq} whose coefficients are given by

$$M_{pq} = \int_{M_\tau} \phi_p \phi_q \, \mathrm{d}vol_\tau.$$

Example 1 *(counterexample to L^2-convergence).* Let $\{M_{\tau,n}\}$ be a sequence of polyhedral surfaces converging to a smooth surface M totally normally. We show that in general $\|\vec{H}_{dis,n} - \vec{H}\|_{L^2}$ does not converge to zero. Consider the cylinder M of height 2π and radius 1. We construct a sequence of a polyhedral surfaces $\{M_{\tau,n}\}$ whose vertices lie on this cylinder and which converges to M totally normally. Let the cylinder be parameterized as

$$x = \cos u, \quad y = \sin u, \quad z = v.$$

Let the vertices of $M_{\tau,n}$ be given by

$$u = \frac{i\pi}{n} \qquad i = 0, ..., 2n-1$$
$$v = \begin{cases} 2j \sin \frac{\pi}{2n} & j = 0, ..., 2n-1 \\ 2\pi & j = 2n \end{cases}$$

This corresponds (up the uppermost layer) to folding along the vertical lines a regular planar quad-grid of edge length

$$h_n = 2 \sin \frac{\pi}{2n}.$$

In other words, all faces of $M_{\tau,n}$ are rectangular (in fact quadratic except for the uppermost layer). It will now depend on the *tessellation pattern* of this quad-

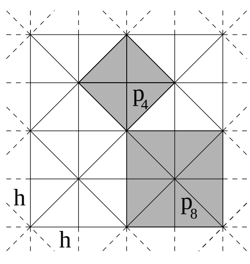

Figure 7: Discrete mean curvature does not converge in L^2 for a $4-8$ tessellation of a regular quad grid, because the ratio between the areas of the stencils of p_4 and p_8 does not converge to 1.

grid whether there is L^2-convergence of discrete mean curvature or not. Indeed, consider the regular $4-8$ tessellation scheme depicted in Figure 7. There are two kinds of vertices - those of valence 4 and those of valence 8. Call them p_4 and p_8, respectively. Let ϕ_{p_4} and ϕ_{p_8} denote the corresponding nodal basis functions. Then by equation (40) the coefficients of the weak mean curvature satisfy

$$\langle \vec{H} | \phi_{p_4} \rangle = \langle \vec{H} | \phi_{p_8} \rangle = -2 \left(1 - \cos \frac{\pi}{n}\right) \cdot \partial_r,$$

where ∂_r denotes the (radial) outward cylinder normal. By the symmetry of the problem there exist constants $a_n, b_n \in \mathbb{R}$ such that

$$\vec{H}_{dis,n} = \sum_{p_4} a_n \cdot \phi_{p_4} \cdot \partial_r + \sum_{p_8} b_n \cdot \phi_{p_8} \cdot \partial_r + \text{boundary contributions.}$$

Set

$$\lambda_n := -\left(1 - \cos \frac{\pi}{n}\right).$$

Using equation (43) one verifies that

$$a_n = 12 \cdot \frac{\lambda_n}{h_n^2} \cdot \frac{4 + \lambda_n}{8 - \lambda_n^2}$$
$$b_n = 12 \cdot \frac{\lambda_n}{h_n^2} \cdot \frac{\lambda_n}{\lambda_n^2 - 8}.$$

Since $\lim_{n\to\infty}(\lambda_n / h_n^2) = -1/2$, it follows that

$$\lim_{n\to\infty} a_n = -3 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0,$$

so that asymptotically *only the vertices of valence* 4 *but not those of valence* 8 *contribute to discrete mean curvature*,

$$\vec{H}_{dis,n} \sim -3 \sum_{p_4} \phi_{p_4} \cdot \partial_r + \text{boundary contributions.}$$

Hence, $\vec{H}_{dis,n}$ is a family of PL-functions oscillating between -3 (at the vertices of valence 4) and 0 (at the vertices of valence 8) with ever growing frequencies. Such a family does not converge in L^2.

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