F3
Mathematical visualization

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Mathematical visualization is the use of pictures to understand and convey mathematics. This has been important – especially in geometry – for thousands of years. With the advent of computers, most mathematical pictures or diagrams these days are drawn not by hand but using computer graphics programs.

We can distinguish several different kinds of mathematical diagrams [21]: topological diagrams (including knot diagrams), 2D geometric diagrams, 2D renderings of 3D objects, or even 3D models.

With computers, we can create not only narrative animations – telling a fixed story by navigating along a chosen path through a parameter space – but also interactive animations, allowing the viewer to make her own explorations. Even at the level of 2D geometric diagrams, modern interactive packages like Cinderella (cinderella.de) have led to new insights into elementary euclidean geometry.

Drawing accurate images of 3D geometric objects by hand requires care and training in perspective. But these are easy to create with computer graphics, and it is trivial even to render a pair of images for stereoscopic viewing with the left and right eye.

In contrast, topological diagrams (including knot diagrams) are easier to draw by hand because of their flexibility, but are harder to draw with a computer because there is no exact given geometry. To find a nice shape for a topological object [22], one approach is to minimize some geometric energy. This is of course closely connected to classical topics in differential geometry, starting with minimal surfaces.

Tools for mathematical visualization have been central to the research in geometry in Berlin for 25 years. Seeing the results of the mathematical experiments we run gives us a deeper understanding of the structures under consideration and thus leads to new conjectures which we can try to prove. But the visualization software almost always works with discrete models, and interesting mathematical problems come up in the process of discretization. Difficulties that arise when implementing a theory on the computer can often shed particular light on areas where our understanding is still incomplete, leading to new insights.

While any discretization of a smooth problem should converge to that original problem in the limit, for many geometric problems there seem to be special discretizations which capture qualitative aspects of the smooth theory even at a very coarse level of discretization. The idea behind the relatively new field of *discrete differential geometry* is to search for these structure-preserving discretizations, which are in a sense even more fundamental than the smooth limit.

Berlin has been a center for the development of discrete differential geometry, as shown for instance by the many workshops we have organized, in Berlin, Oberwolfach [2] and elsewhere.

The visualization projects in MATHEON have cooperated closely with other DFG projects like the Research Group *Polyhedral Surfaces* and the SFB/Transregio *Discretization in Geometry and Dynamics*.

The discrete differential geometry developed in MATHEON projects has found applications in architecture [12, 18]. It has led to new geometric proofs of combinatorial theorems [10]. The first two topics discussed in more detail below – discrete conformal maps and discrete smoke-ring flow – fit directly into this area of discrete differential geometry.

F3—1 Discrete conformal maps

Conformal maps between surfaces are those which preserve angles; this is of course a classical topic closely connected to complex analysis. For many applications, say in computer graphics, we would like to find a *conformal flattening*, that is, a conformal map from a surface in space – given as a triangulation – to a domain in the plane. This is useful, for instance, if we want to apply a (periodic planar) texture map across the surface with as little distortion as possible.

Following a suggestion of Thurston, there is a well-developed theory of discrete Riemann mapping (between planar domains) based on circle packings – or more generally circle patterns [3]. This kind of approach extends nicely to special classes of surfaces in space, like minimal surfaces with their conformal coordinate-line parametrizations. In MATHEON, we developed the theory of discrete S-isothermic surfaces [1], leading to striking pictures of minimal surfaces like that in in Figure 1.

One approach to solve the conformal flattening problem is to use the geometry of a surface

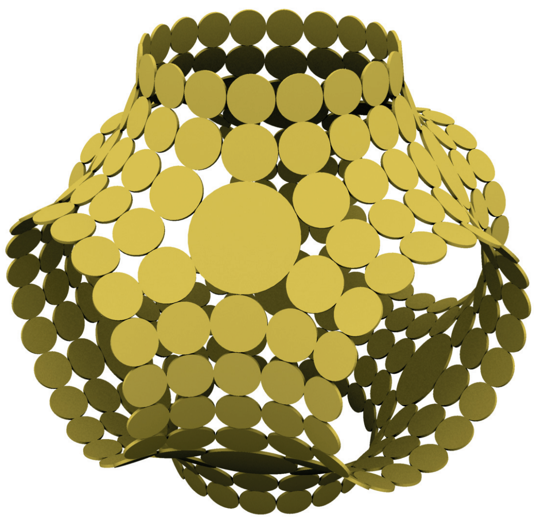

Figure 1. *A discrete S-isothermic version of the Schwarz minimal surface, based on packings of circles and spheres. This image was used as the logo for the DFG Research Group* Polyhedral Surfaces.

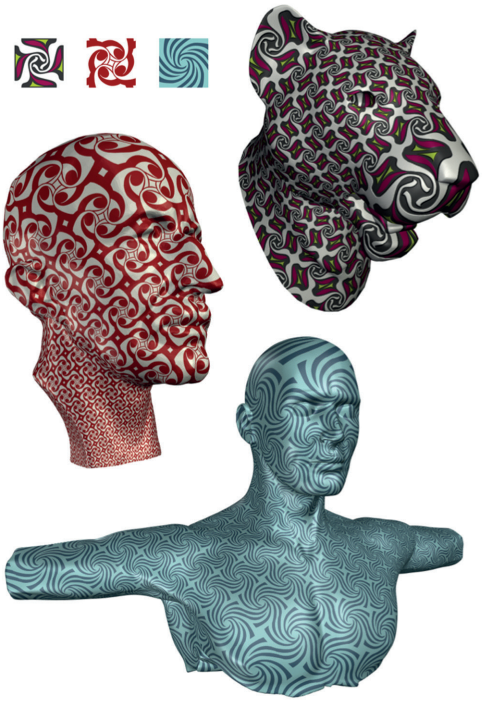

Figure 2. *The new notion of conformal equivalence for triangle meshes gives good conformal flattenings, allowing us to apply a planar texture map in a angle-preserving way.*

mesh to approximate angles for a corresponding circle pattern in the plane [11]. We successfully used this approach, for instance, to conformally map copies of the MATHEON logo onto the MATHEON Buddy Bear, as described in Showcase 19.

More recently we developed a new approach, using a direct and simple notion of conformal equivalence between triangle meshes [19]. This is no longer based directly on the idea of preserving angles, but instead on the idea that a conformal map distorts all lengths by the same local factor, independent of direction. The geometry of a triangle mesh is given by its edge lengths. If we multiply the lengths of all edges incident to a given vertex by a common factor, the resulting mesh is by definition discretely conformally equivalent. This approach leads to a better algorithm for conformal flattening, as illustrated in Figure 2.

F3—2 Discrete smoke rings

Any fluid flow in space can be recovered (using the Biot–Savart law) from its vorticity. In many cases (as seen in smoke rings) this is concentrated along vortex lines. The “localized induction approximation” says that these vortex lines move by the so-called smoke-ring flow,

$$\dot{\gamma} = \kappa B = \gamma' \times \gamma''$$

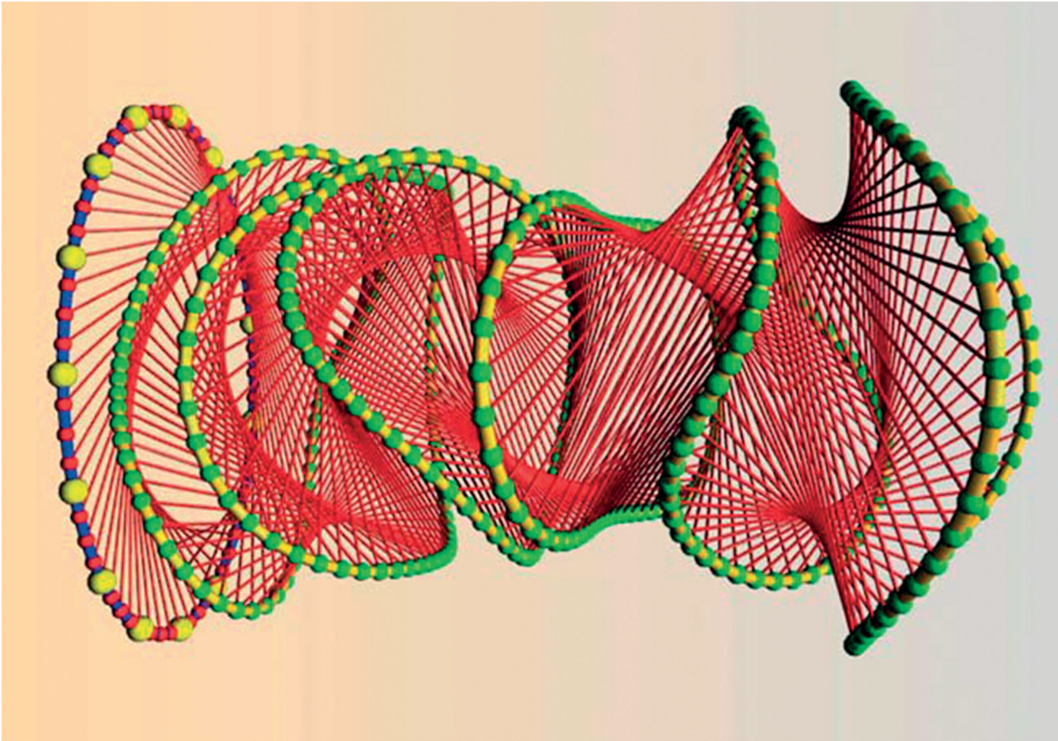

Figure 3. *A doubly discrete version of the smoke-ring flow preserves the integrable structure of the smooth equation and forms the basis for our simulations of smoke.*

an integrable equation preserving length by moving in the binormal direction. The integrable structure led us to a nice structure-preserving discretization, the doubly discrete smoke-ring flow, developed first for equilateral polygons [9] and then for general polygons [14]. A implementation in jReality lets us explore our theory interactively, as in Figure 3.

In MATHEON we have used this theory to develop fast interactive algorithms for fluid flows. We approximate the vorticity by a small number of vortex rings, and let these evolve not merely locally according to the discrete smoke-ring flow, but also according to their global interaction with each other [14]. With a bit more work, we can incorporate obstacles – which shed vortex rings as the fluid flows around them – as in Figure 4. We have also used these techniques to model the motion of objects under water [24].

Computer games and computer graphics movies often incorporate smoke after virtual explosions. Because our complex smoke-like flows are modeled quite simply based on a few vortex rings, desired effects are quite easy to design. We have successfully incorporated our algorithms into software used at DreamWorks Animations. They have been used for animated feature films like the 2010 action comedy *Megamind*, directed by Tom McGrath. One scene – with several towers of smoke rendered with our algorithm – is shown in Figure 5.

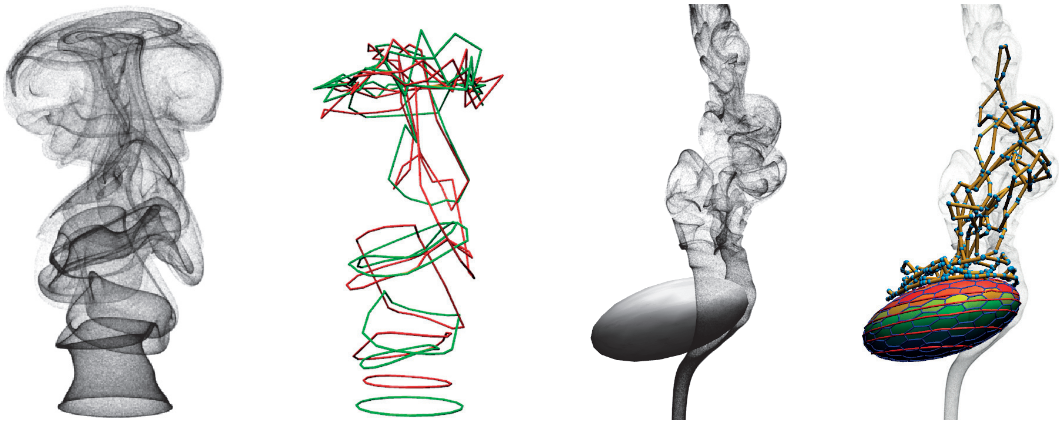

Figure 4. *The smoke jet* (far left) *is modeled by a simple evolution of vortex rings* (left). *We can also model smoke flow* (right) *around an ellipsoidal obstacle, which sheds new vortex rings* (far right).

Figure 5. *A still image from the DreamWorks movie* Megamind*, where the columns of smoke were designed using our discrete smoke-ring simulation*

F3—3 Domain coloring of complex functions

The visualization of complex functions is a classical challenge dating back to the 19th century, when the fundamentals of complex analysis were being developed. Since the graph of a complex function naturally lives in $\mathbb{R}^4$, usually only partial information is shown. For instance, graphs of the real and imaginary parts separately – or of the absolute value – are ordinary surfaces in 3D. In the 1990s, the technique of *domain coloring* was introduced to visualize complex functions as flat, colored images. (We gave a hands-on introduction in [17].)

The idea is simple but striking: a rectangular pixel image represents the domain on which a given complex function is evaluated. A *color scheme* (a map $\mathbb{C} \to$ RGB) is applied to the function value at each point of the domain, giving the color of the corresponding pixel. Figure 6 shows two examples.

Within MATHEON we greatly improved this technique [15]. Whereas earlier color schemes were often simple hue–argument relations, we enriched these by additional features such as grids, semi-transparent blending and highlighting of critical points, leading to high-quality color images. The one in Figure 7 won an honorable mention [16] in the 2011 *International Science and Engineering Visualization Challenge.*

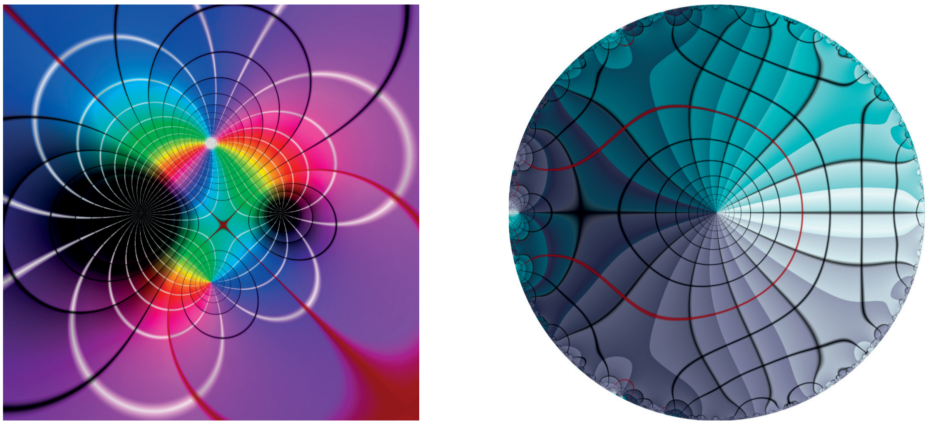

Figure 6. *Domain colorings of the rational function* $\frac{(z+1)^2(z-1)}{(z-i)^2(z+i)}$ (left) *and the first* 50 *terms of the Fabry series* $\sum_{n=0}^{\infty} z^{n!}$ *inside the unit disc* (right)

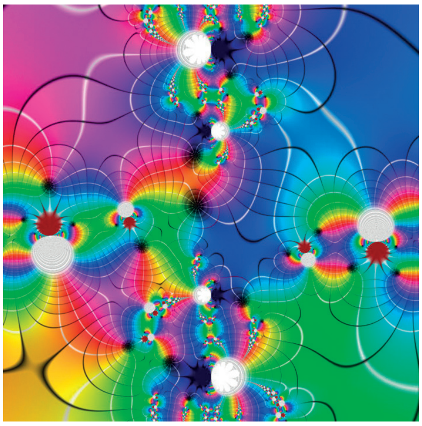

Figure 7. *In this prize-winning image based on domain coloring, the hue gives the argument of a complex number. White regions show poles of the function, while black regions show zeros.*

Of course many complex functions, like the square root or logarithm, are not single-valued on the complex plane. Instead, it is best to consider them as analytic functions on some Riemann surface, a branched cover of the plane. Thus we also "lifted" the coloring technique to 3D models [13] of such Riemann surfaces, as in Figure 8. This qualitative change of paradigm helps us visualize the analytic continuation of complex functions that exhibit branching behavior.

F3—4 Geometric three-manifolds

Bill Thurston's geometrization conjecture, proved by Perelman, says that any three-manifold can be cut into pieces with natural geometric structures – in most cases modeled on hyperbolic

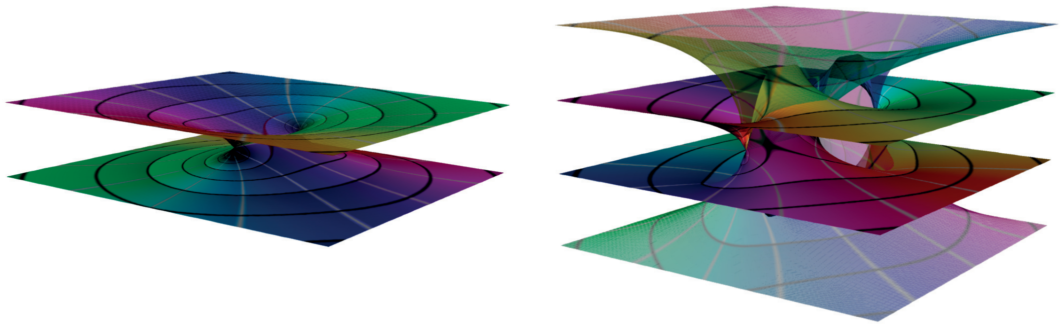

Figure 8. *Domain coloring lifted onto Riemann surface models with branching of order* 2 (left) *and* 4 (right). *Whereas the underlying functions restricted to a single sheet would be discontinuous, the version on the Riemann surface has a globally continuous coloring.*

space. In conjunction with this, Thurston suggested that the best way to visualize a geometric 3-manifold M is by thinking of it not as some small object we might hold in our hands, but as a large space in which we could live. Light rays would follow the geodesics in the geometric structure, and because these geodesics wrap around any loops in M, the pictures are the same as if we were in the universal cover. What we typically see is thus a periodic pattern filling spherical or hyperbolic space.

In MATHEON Charles Gunn has implemented the Maniview package as part of jReality. It takes advantage of the fact that the 4×4 projective matrices implemented by modern graphics chips can represent spherical or hyperbolic motions just as easily as euclidean ones. Thus we get high-performance real-time interactive images in of all of these geometries. Figures 9 and 10 show what it looks like to live in hyperbolic and spherical manifolds. One special challenge is to find reasonable lighting models. For understanding spherical scenes, the new visualization technique of *conformal curvilinear perspective* [6] is often even more useful than the internal view.

F3—5 Virtual reality, jReality and outreach

In MATHEON we have developed the java-based software package jReality [23] for mathematical visualization. One feature is the clear separation between (1) the front end representation of the scene graph, (2) the back end which renders this on a particular device and (3) the tools enabling user-interaction. Thus a jReality program can easily be ported between different environments, including virtual reality with different forms of user interaction. Several features in jReality, including those highlighted in Figure 11, are designed especially for ease of use in virtual reality. (Many of the images in earlier subsections were also rendered in jReality.)

At TU Berlin we have built an immersive virtual-reality theater, the Portal, shown in Figure 12. It has stereoscopic projection on three walls and optical head tracking. We have also installed stereo projection equipment in various lecture halls including the Audimax (Figure 13), which seats over 1000 people.

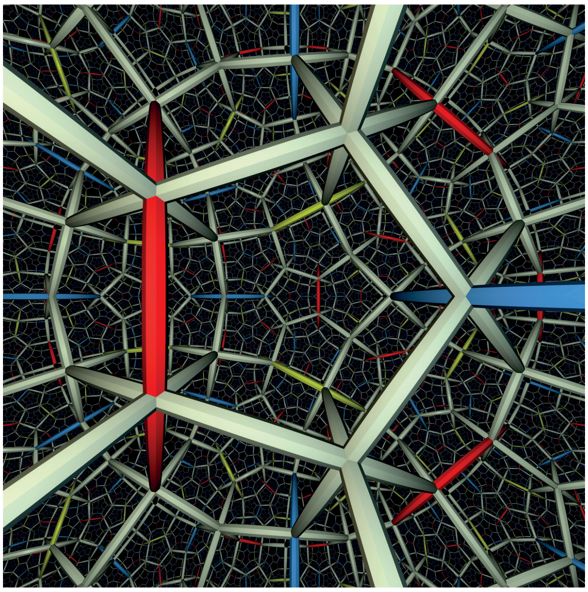

Figure 9. *A view inside a hyperbolic* 3*-manifold – the cover of the three-sphere with* 4*-fold branching over the Borromean rings – rendered in jReality with Maniview*

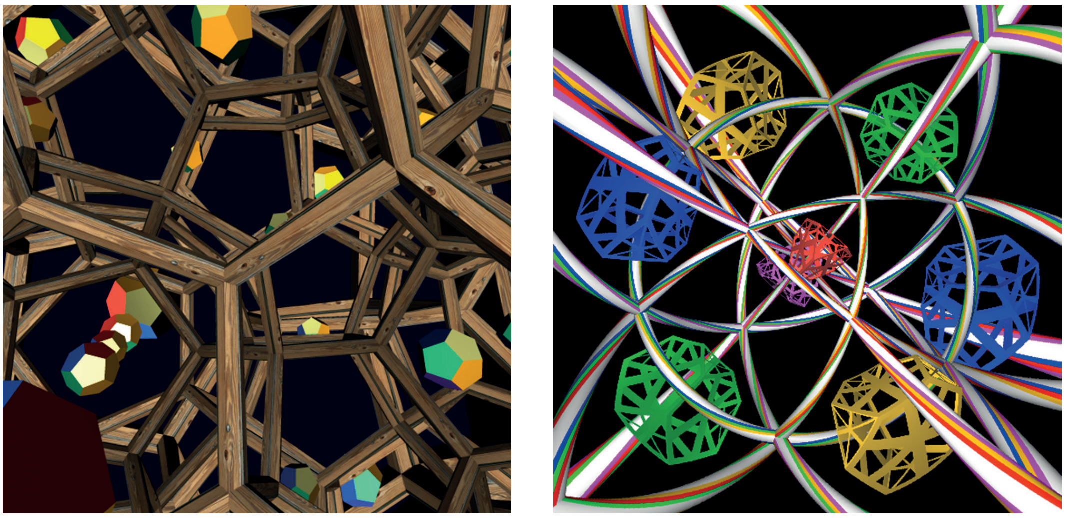

Figure 10. Left: *A view inside a spherical* 3*-manifold, the Poincaré homology sphere, whose universal cover tiles the three-sphere with 120 dodecahedra.* Right: *Another symmetric tiling of* $\mathbb{S}^3$ *– by ten truncated tetrahedra – rendered instead with curvilinear perspective*

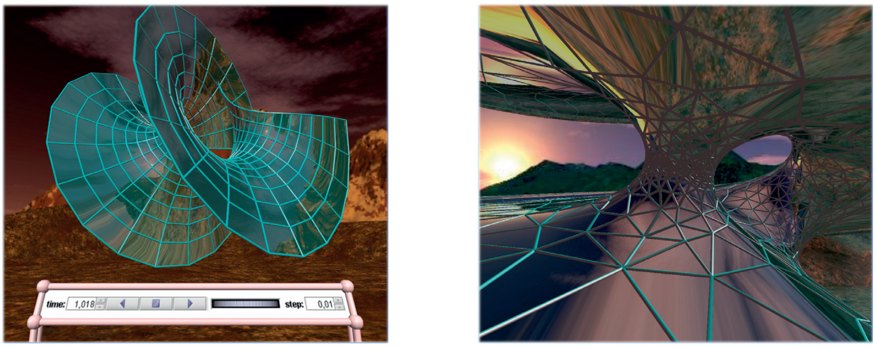

Figure 11. *In jReality, panels for user controls can be integrated into the 3D scene* (left). *One mode of interaction lets the user walk on a curved surface* (right), *automatically following its height.*

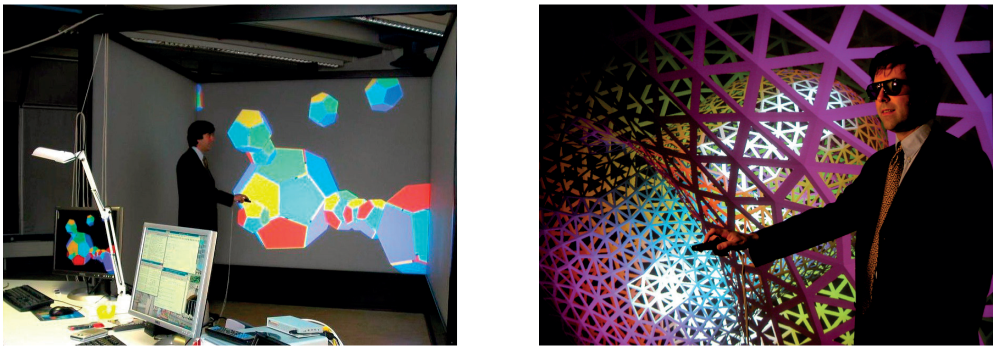

Figure 12. *The Portal* (left) *has three walls, each with stereoscopic projection. User interaction* (right) *is through tracked glasses and a hand-held wand.*

Figure 13. *A mathematical animation in 3D can be shown to over a thousand students in the Audimax at TU Berlin*

Figure 14. *At the reopening ceremony for the* Mathematisches Kabinett *in Munich, several of our exhibits based on jReality were visible.*

Using jReality, MATHEON has conducted many outreach activities. For instance, starting during the German "Year of Mathematics 2008", we have been active contributors to the *Imaginary* project, headed in Oberwolfach, which develops open-source mathematical exhibits. We have installed several exhibits in the redesigned *Mathematisches Kabinett* at the Deutsches Museum in Munich (see Figure 14) and also at the new MiMa in Oberwolfach.

With a design based on the tight form [4, 5] of the Borromean rings, we won the contest to design a logo (shown in Figure 15) for the International Mathematical Union. Our short video [7, 8] about the mathematics behind this logo was shown at the opening and closing ceremonies of the International Congress of Mathematicians (ICM 2006) in Madrid. It was produced in jReality, using a back end that drives Pixar's Renderman for high-quality images.

We have contributed to various exhibits of mathematical art – for instance at the Institut Henri Poincaré in Paris – contributing not only computer graphics prints, but also sculptures produced from computer models on a 3D printer at TU Berlin. The "Minimal Flowers" [20] shown in Figure 16 are based on thickened minimal surfaces spanning knotted boundary curves with various orders of rotational symmetry.

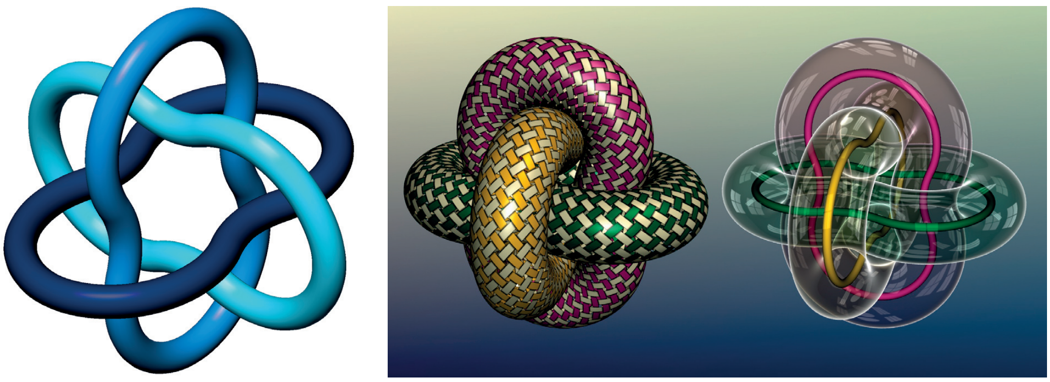

Figure 15. *Our logo for the IMU* (left) *was the subject of a short video* (right) *about the Borromean rings. It was produced with the Renderman back end for jReality, using custom shaders for rope textures and soap films.*

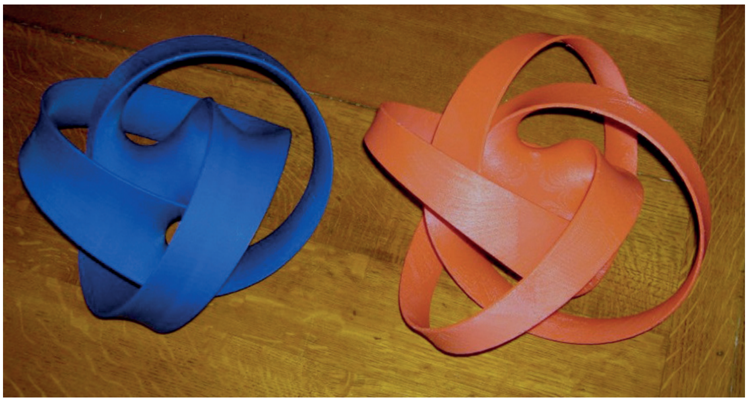

Figure 16. *Sullivan's sculptures "Minimal Flower 3" and "Minimal Flower 4" are based on minimal surfaces spanning knots with three- and four-fold symmetry, and are printed on a 3D printer.*

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