Chapter 1
Realization of Regular Maps of Large Genus

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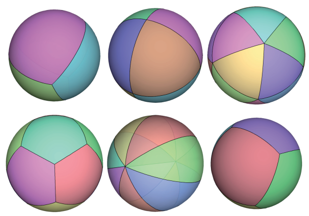
Abstract Regular map is an algebraic concept to describe most symmetric tilings of closed surfaces of arbitrary genus. All regular maps resp. symmetric tilings of surfaces up to genus 302 are algebraically known in the form of symmetry groups acting on their universal covering spaces. But still little is known about geometric realizations, i.e. finding most symmetric embeddings of closed surfaces and a supported most symmetric tiling. In this report, we will construct some new highly symmetric embeddings of regular maps of up to genus 61 and thereby shed some new light on this fundamental problem at the interface of algebra, differential geometry, and topology.

1.1 Introduction

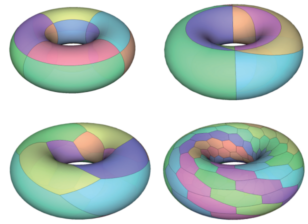
Tiling of closed surfaces into non-overlapping faces is one of the central topics in surface topology and computer graphics. Either the surface is given and a nice tiling of this surface has to be found or the tiling is given and the surface on which the tiling is the most symmetric has to be found. This paper explores the later case but restricts the tiling scheme to the class of regular maps.

The concept of map was first introduced by Coxeter and Moser [2]. A *map* is a family of polygonal faces such that any two faces share an edge or vertex, or are disconnected. Each edge of the maps belongs precisely to two faces, the faces containing a given vertex form a single cycle of adjacent faces and between any two faces is a chain of adjacent faces. In other words, it is a closed 2-manifold without boundaries obtained by glueing topologically equivalent polygonal faces. If the maps has p-gonal faces and q-gonal vertex-figures (number of faces around at a vertex), then it is has the Schläfi symbol $\{p,q\}$.

A *regular map* is a map which is flag transitive. This means that, on the surface, if a vertex or an edge or a face is mapped to another vertex or an edge or a face, then the map is one to one and preserves all adjacency properties between the vertices, edges and faces. Regular maps can be viewed as generalization of the Platonic solids into



(a) Spherical regular maps



(b) Genus 1 regular maps

Fig. 1.1 Example of genus 0 and genus 1 regular maps, where the first set corresponds to the Platonic solids and the second set to tilings of the torus.

higher genus surfaces. They define regular tilings of closed surfaces and their group structure can be used to move along the surface which behaves like an “hyperbolic” parameterization.

Figure 1.1 illustrates some examples of low genus regular maps which include the Platonic solids, the Hosohedron and some tilings of the torus.

In this report, we give realizations of regular maps of genus > 20 that we classify to be *large* genus regular maps. In Figure 1.2 is an example of a genus 61 regular map which is the highest genus regular map ever visualized so far. Our main contribution is the targetless tubification described in Section 1.4.2. All the images generated in this paper are produced by our algorithm.

The chapter is organized as follows. First, we will give theoretical backgrounds on regular maps including geometric and algebraic characterization. Second, we will describe the basic procedure to obtain, with maximal symmetry, large genus surfaces. And finally, we will give a description of the method used to produce tilings on these large genus surfaces.

1.2 Related Works

Up to now, there is no general method to visualize regular maps but a lot is already known about their symmetry group, see for example Conder [1]. The problem is two-fold, understanding the symmetry group of the regular map and finding a suitable space model for it. Jack van Wijk [5], in his Siggraph paper, suggested a generic approach which gives interesting visualization of some of the lower genus regular maps up to genus 29. He succeed to handle about 50 cases by using a brute force computer search. However, his method is too restrictive and cannot realize even some of the simplest cases. Séquin’s investigations ([10], [11], [12]) are also a huge source of inspiration. He uses physical modeling techniques, including sketches, paper models and Styrofoams to finally obtain a computer generated model. Some cases have been solved by his method from genus 2 to genus 5 but each regular map is handled separately. Sequin’s approach are useful for a better understanding of the structure of regular maps but too primitive to handle the large ones. In our

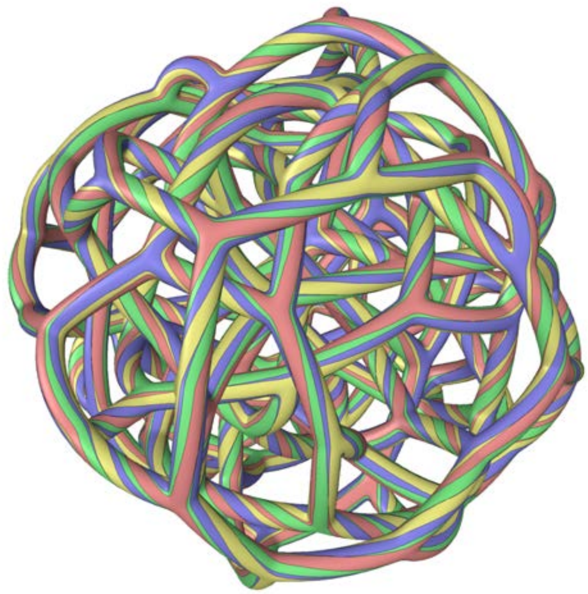


Fig. 1.2 Visualization of a genus 61 surface tiled with 480 hexagons following the regularity of the map R61.1$'\{6,4\}$.

early work [7, 8], We uses the same approach as van Wijk but we added a relaxation procedure to obtain more symmetrical and smooth tubular geometry. We use this relaxation scheme as a second step of the targetless tubification algorithm.

In this paper, we aim at surfaces having more than two junctions and with rich structures to accommodate regular maps. We are not interested in the hosohedral kind of surface. These are the surfaces obtained by taking the tubular neighborhood of Hosohedra.

1.3 Background Notions

1.3.1 On Standard Geometry

Isometric realization of tilings depends on the ambient space where they are embedded. These are: the Sphere, the Euclidean plane and the Hyperbolic plane. For spherical and hyperbolic geometry, we use [9]. Examples of spherical isometric tilings are the Platonic solids. The Euclidean plane can be isometrically tiled by checker-board patterns and Honey-comb like structure. The Hyperbolic plane are tiled by p-gons for large p's. A closed 3D realization of a sub-tiling of the Euclidean plane is a Torus (see Figure 1.1). The closeness of the torus is topologically derived from a parallelogram in the Euclidean plane wrapped in 3D by identifying opposite sides. In this 3D realization, isometry is lost but the topology of the tiling is still preserved. We then only talk about *combinatorial* or *topological* transitivity.

Similar 3D realizations can also be done from the Hyperbolic plane. A genus $g > 1$ surface is derived by taking a $4g$-gon in the Hyperbolic plane and identifying pairwise edges. Hence, any tiling of the Hyperbolic plane can be realized as 3D surfaces by finding a proper $4g$-gon partitioning of this tiling with the correct identification at the boundary. Special case of these tilings are regular maps.

If a map has V vertices, E edges and F faces, then its genus g is given by

$$g = (2-\chi)/2, \tag{1.1}$$

where $\chi = V - E + F$ is the Euler-Poincaré characteristic. It is a property of the surface, independent of the map; the dual map has also the same Euler-Poincaré characteristic χ. Intuitively, the genus of a surface is the number of tunnel in this surface. Depending on their genus, regular maps can be abstractly realized as quotients of spherical tilings, euclidean tilings or hyperbolic tilings.

1.3.2 On Regular Map

A *finitely generated group* is a group of the form $\langle \mathscr{G} | \mathscr{R} \rangle$, where $\mathscr{G}$ is a set of generators and $\mathscr{R}$ is a set of relations. If $R_i \in \mathscr{R}$, then $R_i = I$ which is the identity of the group.

A regular map is a finitely generated group of the following form

$$\mathrm{Sym}(M_S) = \left\langle R,S,T | R^p, S^q, T^2, (RS)^2, (ST)^2, (RT)^2, \mathscr{R}_1, \ldots, \mathscr{R}_m \right\rangle, \tag{1.2}$$

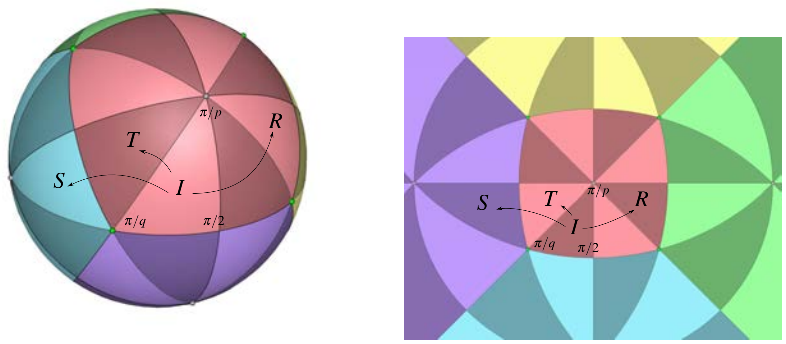
where R is a rotation of $2\pi/q$; S is a rotation of $2\pi/p$ and T is a reflection. They are transformations acting on a fundamental triangle with corner angles $\pi/p, \pi/q$ and $\pi/2$ (see Figure 1.3). Depending on p and q, they can be euclidean motions, special orthogonal matrices (for spherical) or Moëbius transformations (for hyperbolic). $\mathscr{R}_1, \ldots, \mathscr{R}_m$ are extra relations making the group finite. The expression 1.2 is called the symmetry group of the map. It is the set of all automorphisms of the regular map [2].

The symmetry group of the cube (a regular map of type $\{4,3\}$) is defined by

$$\mathrm{Sym}(\mathrm{Cube}) = \left\langle R,S,T | R^4, S^3, T^2, (RS)^2, (ST)^2, (RT)^2 \right\rangle, \tag{1.3}$$

Sym(Cube) can be realized on a blown up Cube, taking as fundamental triangle a spherical triangle with corner angles $\pi/4, \pi/3$ and $\pi/2$ (figure 1.3-a). It can also be visualized as a 2D surface using stereographic projection which is only conformal (Figure 1.3.b) but not isometric.

Orientable regular maps are denoted in Conder [1] by R$g.i\{p,q\}$, which is the $ith-$reflexible orientable map of genus g. $\{p,q\}$ is the Schläfi symbol of the tiling.



(a) (b)

Fig. 1.3 Two different representations of the Cube using: (a) a Sphere and (b) stereographic projection into the plane.

Reflexible means that the transformation T in Equation 1.2 is also an automorphism of the map. Analogously, the dual map is represented by R$g.i'\{q,p\}$. Conder [1] listed all reflixible regular maps of genus 2 to 302. They are given as symmetry groups and used as input to our algorithm.

1.4 Generating Large Genus Surfaces

In this section, we explore in depth techniques to generate and visualize large genus surfaces. Our aim is not only to generate some genus g surface but also a surface with rich topological structure and nice looking shape.

1.4.1 Tubification Process

A genus g surface can be generated by taking a sphere and drill non intersecting g tunnels on it. Another approach, very used for teaching, is the sphere with g handles. It consists mainly of taking Tori and glueing them on a Sphere to form handles. It is then unclear where the Tori should be placed and if the resulting surface can be used to visualize symmetric tilings.

A better approach is the use of a *tubification* process. It consists of taking a tiling of a surface, turning its edges into tubes, its vertices into junctions and its faces into tunnels. For example, a genus 2 surface can be derived from a tubified Hosohedron $\{2,3\}$ as illustrated in Figure 1.4.b. Surfaces with rich structure or regular surfaces can be derived by taking regular tilings having more than two vertices. The Platonic solids are direct examples of these. Even more, we can take any regular maps and

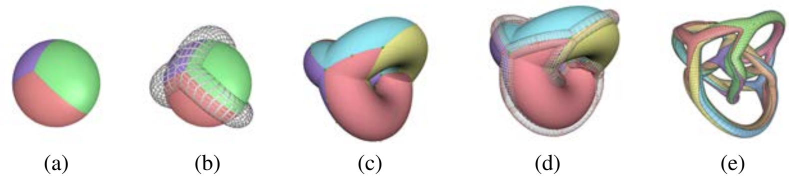


Fig. 1.4 Recurssive tubification starting from: (a) hosohedron $\{2,3\}$, (b) tubification of its edge graph, (c) regular map R2.8$'\{8,3\}$, (d) tubification of its edge graph and (e) regular map R9.3$'\{6,4\}$.

apply the tubification process to derive large genus surfaces. In Figure 1.4.c-d, we show an example of a tubification of the regular map R2.8'$\{8,3\}$.

As in [5], a pairing of source and target map is used to generate higher genus surfaces. The source map is the actual regular map that we want to realize and the target map is the regular map which after tubification gives a space model for the source map. More precisely, let $(M_i)_{i\in I}$ be a finite sequence of regular maps such that a space model of M_{i+1} is the tubification of M_i. For a given n, if for all $i < n$, the M_i's are realized, then a tubification of M_{n-1} is a space model of M_n. Otherwise, we cannot give a space model for M_n. This become now a classical method used to visualize successfully large class of regular maps. In the next section, we show that, in fact, the sequence of M_i's is not needed, only the pairing of source-target map is enough.

1.4.2 Targetless tubification

The tubification of an existing regular map has a critical issue since it needs an actual realization of the target regular map. Hence, if the target regular map does not have a 3D embedding, then the tubification cannot be applied and thus no higher genus surface is generated.

We give a solution to this restriction by taking advantage of the planar representation of the target regular map. We call the process a *targetless tubification*. Targetless in the sense that no actual 3D embedding of the target regular map is needed but only an embedding of its edge graph is sufficient.

We generalize the torus case to Hyperbolic space. More precisely, suppose we have a tiling of a flat torus with its 2D edge graph. This edge graph can be mapped to 3D using the usual parameterization function and hence a tubular surface is derived naturally. In this process, only the edge graph is needed to be embedded in space, not the 2D tiling. We do also the same process in Hyperbolic space but since we do not have an explicit parameterization, we do as follows: first, we identify explicitly boundary edges of the map and second, we apply the constrained relaxation procedure described in [7] to get symmetry and smoothness. All the higher genus surfaces in this paper were produced by this simple procedure. An example of a genus 5 surface obtained by the edge graph of the regular map R2.4$\{5,10\}$ is illustrated in Figure 1.5. In this illustration, we start with the hyperbolic realization of R2.4 with the identification at the boundary (represented by the arrows). We then match the boundary edges having the same label, head to head and tail to tail. This results in a 2D connected graph which has the same combinatorics has the edge graph of the underlying regular map. This 2D graph is then smoothed using spring energy.

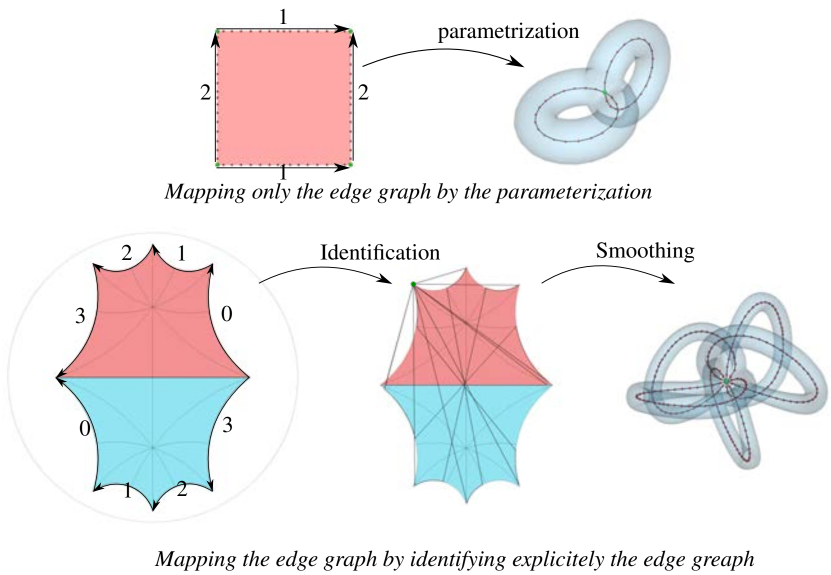


Fig. 1.5 Construction of a high genus surface by embedding directly the edge graph of the target regular map.

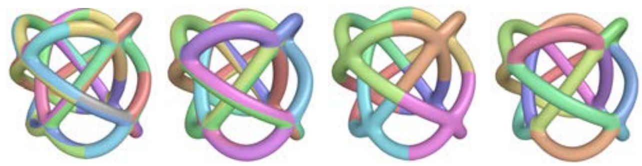
Notice that no actual embedding of the regular map R2.4 is needed (see [12] for a 3D realization of this map). Our technique can be applied to any planar representation of a regular map to generate a 3D tubular surface obtained from its edge graph. Below are some examples of large genus regular map generated by the above method.

1.5 Topological Group Structure

In this section, we define a topological group structure on the surface generated previously that we denote S_g.

1.5.1 Partition by Tube Elements

The recursive tubfication procedure derived in section 1.4.1 allows us to choose a tiling of S_g. We have for example a tiling with quarter-tubes, with half-tubes, with tube junctions, with full tubes or with multiple quarter-tubes (see Figure 1.6). We call one of these a *fundamental domain* of S_g, and as for every group, we can cover the surface by copies of this fundamental patch. These tilings are induced naturally from the underlying regular map used to derive the tubular surface.



(a) quarter-tubes (b) half-tubes (c) tube junctions (c) full tubes

Fig. 1.6 Example of a partition of S_g with some elements of the tubes.

The next step is now to define a group structure induced by the tube element in order to define a parameterization of S_g. This parameterization will be then used to map other regular maps as described in Section 1.6.

1.5.2 Deriving the Symmetry Group

We restrict our construction to the case of a tiling with quarter-tubes as in [5]. The other cases can be handled analogously.

Let $\mathscr{Q}$ be the set containing all colored quarter-tubes of S_g. We label the edge of a quarter-tube by a, b, c and d, where a is the one at the junction and b, c, d are the next counter-clockwise edges. The orientation is defined by the normal of the surface at each quarter-tube.

We define a basic operation Adj_x on $\mathscr{Q}$ which takes a quarter-tube Q and returns the quarter-tube adjacent to Q at edge x:

$$\begin{aligned} \mathrm{Adj}_x : \mathscr{Q} &\to \mathscr{Q} \\ Q &\mapsto \text{quartertube adjacent to } Q \text{ at edge } x \end{aligned}$$

For example, $(\mathrm{Adj}_b)^2$ is the identity since making two quarter-tube steps around a tube get back to the start. $(\mathrm{Adj}_a)^2$ is also the identity. Let Q_I be a fundamental domain of $\mathscr{Q}$ (it can be any quarter-tube of $\mathscr{Q}$). We define three operations A, B and C on $\mathscr{Q}$ as follows:

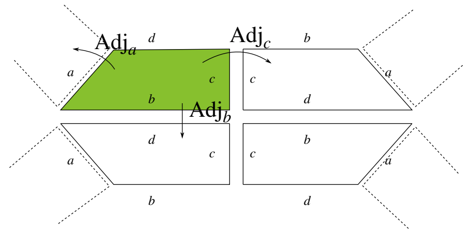


Fig. 1.7 The adjacency operator defined on the set of quarter-tubes.

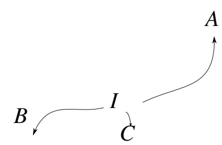


Fig. 1.8 Three adjacency operators acting on the quarter-tube tiling: A shifts Q_I two positions positively around a hole; B rotates Q_I around the junction; C shifts Q_I one position down.

- A shifts Q_I two positions positively around a hole, more precisely $Q_A = \mathrm{Adj}_a(\mathrm{Adj}_c(Q_I))$;
- B rotates Q_I around the junction, $Q_B = \mathrm{Adj}_d(\mathrm{Adj}_a(Q_I))$;
- C shifts Q_I one position down, $Q_C = \mathrm{Adj}_b(Q_I)$.

Here, Q_M denotes the quarter-tube obtained by applying a transformation M to Q_I.

We can see A as a transformation moving a tube around a hole, B switches from one hole to another hole and C enables to reconstruct a full tube from a quarter of a tube. Using Adj_x , we can derive the following relation

$$(CBA)^2 = (BA)^2 = (CB)^2 = I$$

where, I denotes the identity transformation. Using the underlying symmetry group of the tiling used to build S_g we can define a symmetry group of S_g as

$$\begin{aligned}\mathrm{Sym}(S_g) = \big\langle A,B,C\,|\,&A^{p_t},B^{q_t},C^2,(CB)^2,(BA)^2,(CBA)^2,\\ &g_1(A,B,CB),\ldots,g_n(A,B,CB)\big\rangle\,,\end{aligned}$$

where, the g_i's are the extra relations of the symmetry group of the underlying regular map.

A group structure on the genus 5 surface shown in Figure 1.8 is given by

$$\mathrm{Sym}(M_5) = \big\langle A,B,C\,|\,A^4,B^3,C^2,(BA)^2,(CB)^2,(CBA)^2\big\rangle\,.$$

This group has exactly 12×4 quarter-tubes highlighted in Figure 1.8. Once the group structure is introduced on S_g, we can unfold this surface in hyperbolic space to embed a regular map on it.

1.6 Tilings

1.6.1 Matching Symmetry Groups

In this section, we give a brief description of the use of the symmetry group introduced on S_g to realize a regular map. The regular map is defined with its symmetry group $\mathrm{Sym}(S_{\mathrm{map}})$ realized as planar tiling in Hyperbolic space.

The first step is to make an hyperbolic parameterization of S_g. This process is similar to the torus case (space model for genus 1 regular map) where the parameterization is done onto the unit square. For high genus surfaces, the parameterization is done by choosing a suitable fundamental quadrilateral hQ_I in hyperbolic space and set it as fundamental domain of $\mathrm{Sym}(S_g)$. The idea here is to make a 2D realization of $\mathrm{Sym}(S_g)$ using another fundamental domain. Once the parameterization is done, the regular map can be naturally mapped using the inverse mapping. An overview of the algorithm is illustrated in Figure 1.9. The remaining problem is then on the construction of hQ_i and the hyperbolic transformations corresponding to the elements of $\mathrm{Sym}(S_g)$.

The construction of hQ_I depends on the matching between $\mathrm{Sym}(S_g)$ and $\mathrm{Sym}(S_{\mathrm{map}})$. These matchings are heuristics which check if there exists a partition of $\mathrm{Sym}(S_{\mathrm{map}})$ by $\mathrm{Sym}(S_g)$. A necessary condition is that the order of $\mathrm{Sym}(S_g)$ should dives the order of $\mathrm{Sym}(S_{\mathrm{map}})$ and a sufficient condition is the existence of a subgroup of $\mathrm{Sym}(S_{\mathrm{map}})$. In the successful case, hQ_I can be constructed, otherwise S_g is not a suitable space model for $\mathrm{Sym}(S_{\mathrm{map}})$ and the mapping cannot be done. Matchings between regular maps are generated using van Wijk [5] heuristics. It consists of a pairing of source map and target map where the second is a lower genus regular map which after tubification gives a space model the first. His heuristic also provides the exact position of the four points of hQ_I in hyperbolic space.

In his lists, there are several mappings which cannot be visualized since the target map does not have a 3D realization. This is mainly the case for the large genus regular maps which depend on several low genus ones. These cases are handled by the targetless tubification in Section 1.4.1.

1.7 Geometric Construction

In this section, we assume that the target map has a 3D realization and describe the geometrical tools used to construct the tubes. As stated in Section 1.4, the tubes

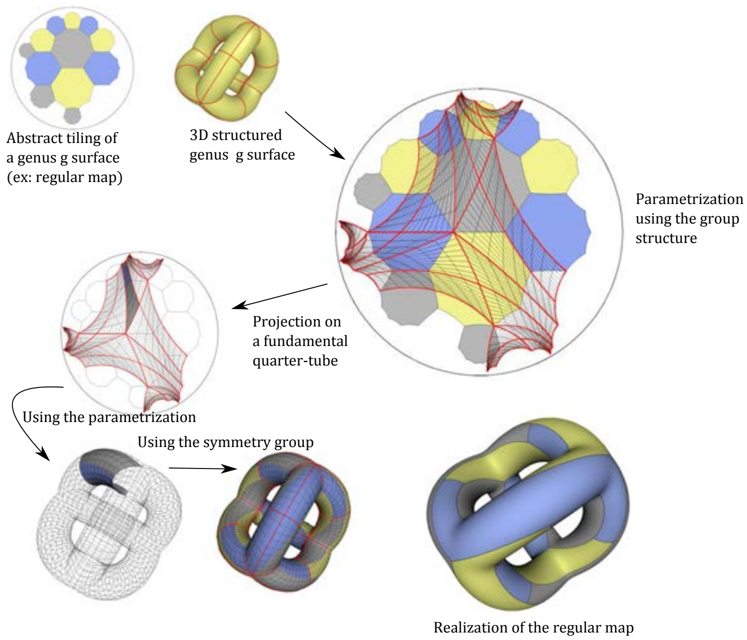
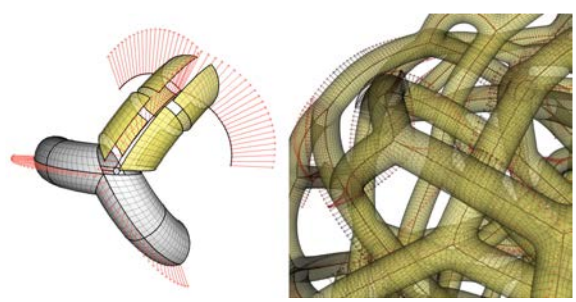


Fig. 1.9 Pipeline to visualize a regular map on a structured genus g surface.

are derived from the edges of an existing regular map. These edges define a control skeleton $\mathscr{S}$ of the tubular surface and is used to deform the shape of the tube.

The normals along $\mathscr{S}$ are induced from the surface. Using these normals, we construct the tube from four quarter tubes, highlighted in Figure 1.10. Each quarter tube consists of half ellipses connected to each other which starts at the center of an edge and rotates from 0 to $\phi/2$ when approaching a junction, where ϕ is the angle with the adjacent edge at the junction. This is illustrated in Figure 1.10.

Fig. 1.10 Generating tubes from quarter tubes. The red arrow represent the normals along the edge of the tiling, induced from the surface (left). At the junction, each quarter tube meets with the correct angle and then identified to form a connected surface (right).



We did not write the mathematical construction of the tubes here, this is extensively studied in [7]. For the targetless tubification, the normals are obtained from the local orientation at the node junctions. These local orientations are derived from the group structure of the underlying regular map which tells us that it must be an orientable surface and hence the normals along its edge-graph must be continuous (take a normal and parallel transport it along any cycle of the graph, then it should give the same normal when it comes back). Normals along an edge are generated by interpolating the two normals at the two endpoints of its junction.

For a better smoothness at the junctions, we do a Catmull-Clark smoothing [6]. New points are not inserted but are only used as mask to relax the points. This is similar to a laplacian smoothing for quad meshes.

1.8 Examples of High Genus Regular Map

In Figure 1.11, few examples of large genus regular maps generated by our targetless tubification are illustrated. The arrows are the matchings between source and target map. These are found using the heuristics presented in [5]. We choose especially large genus regular maps which are closed to spherical and euclidean tilings. Namely, maps for which the integer distance between p and q is not so big.

We succeed to generate all the regular maps missing in [5] which are more than two-fold increases of the current results. In this paper, we emphasis on the visualization of large genus regular maps with self-intersection free even for very high genus surfaces.

The choice of the tube radius is crucial in this process but it is closely related to the spring energy parameter (attraction and repulsion). Hence, we leave it interactive and modified by visual inspection. In all of our experiment, only few adjustment is needed to have a non self-intersecting surface.

1.9 Conclusion

We presented a method to generate large genus regular maps. Regular maps are generalization of the Platonic solids into higher genus surface. These are realized by using a new targetless tubification procedure which does not require any actual embedding of a target shape to generate a genus g surface.

Regular map is an intriguing surface and having a nice visualization of them remains an interesting and unsolved problem. So far, we did not find any practical application of those shapes if not for symmetric tiling of closed surface. They can also be good models used for teaching and understanding how symmetry group work.

Similar to the Platonic solids, regular maps are the most symmetric tiling we can use for high genus surfaces but we need to find the correct space model where these symmetries can be appreciated.

What is not described in this paper is an automatic algorithm which gives the identified edges in the planar representation of the target maps. We will leave this detail as future work.

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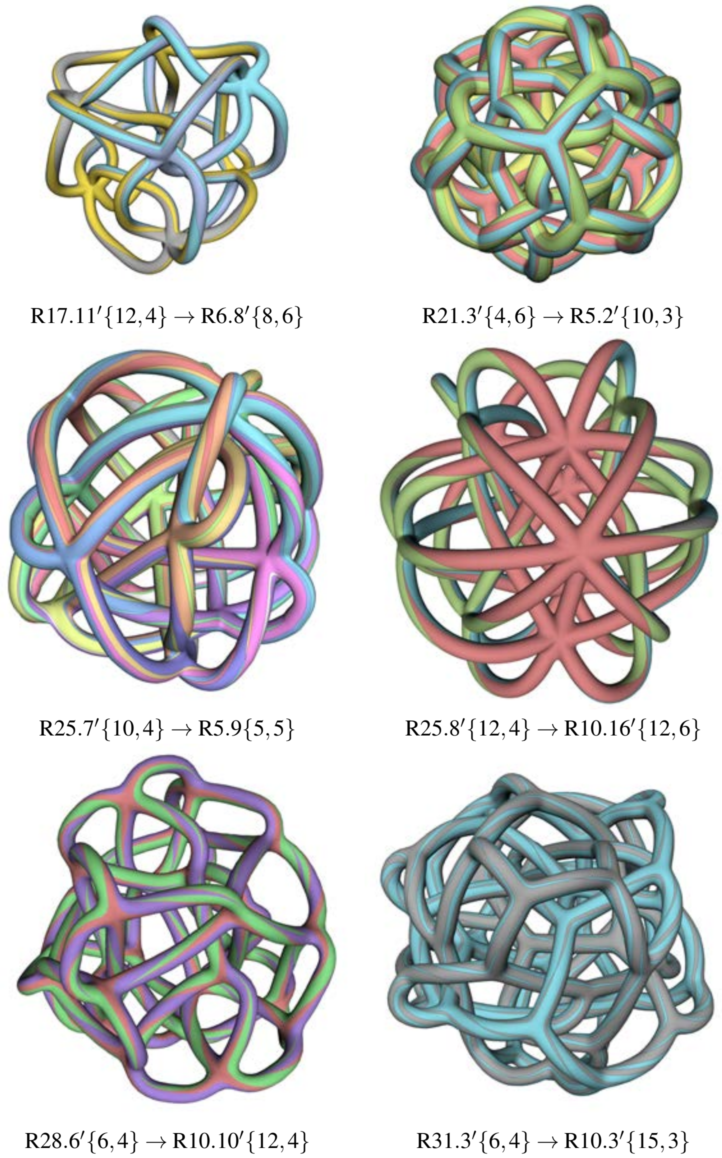

Fig. 1.11 Some large genus regular maps generated by the targetless method.

References

1. Conder M,Regular Maps, August 2012,
   http://www.math.auckland.ac.nz/ conder/OrientableRegularMaps301.txt.
2. Coxeter H. and Moser W., Generators and Relations for Discrete Groups, *Springer Verlag*, 1957.
3. Dyn N., Levin D. and Gregory J. A., A 4-point Interpolatory Subdivision Scheme for Curve Design, *Computer Aided Geometric Design*, 4:257-268, 1987.
4. Levy S., The Eightfold Way: The Beauty of Klein's Quartic Curve, *Cambridge University Press*, 1999.
5. van Wijk J. J., Symmetric Tiling of Closed Surfaces: Visualization of Regular Maps. *Proc. SIGGRAPH 2009*, 49:1-12, 2009.
6. van Wijk J. J. and Cohen A. M., Visualization of Seifert Surfaces, *IEEE Transactions on Visualization and Computer Graphics 12*, 4:485-496, 2006.
7. Razafindrazaka F., Visualization of High Genus Regular Maps, master's thesis, *Freie Universität Berlin*, 2012.
8. Razafindrazaka F. and Polthier K., Regular map smoothing, DGCI 2013, Seville, 2013.
9. Reid M. and Szendro B., Geometry and Topology, *Cambridge University Press*, 2005.
10. Séquin C. H., Patterns on the Genus-3 Klein quartic, *In Proc. BRIDGES 2006 Conference*, London, 245-254.
11. Séquin C. H., Symmetric Embedding of Locally Regular Hyperbolic Tilings, *In Proc. BRIDGES 2007 Conference*, San Sebastian, 379-388.
12. Séquin C. H., My Search for Symmetrical Embeddings of Regular Maps, *In Proc. BRIDGES 2010 Conference*, Hungary, 85-94.
13. William S. Massey, A Basic Course in Algebraic Topology, *Graduate Texts in Mathematics*, Springer, 1991.