Complex Polynomial Mandalas and their Symmetries

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Abstract

We present an application of the classical Schwarz reflection principle to create *complex mandalas*—symmetric shapes resulting from the transformation of simple curves by complex polynomials—and give various illustrations of how their symmetry relates to the polynomials' set of zeros. Finally we use the winding numbers inside the segments enclosed by the transformed curves to obtain fully coloured patterns in the spirit of many mandalas found in real-life.

Introduction

The buddhistic and hinduistic *Mandala* symbols have inspired art for centuries through the harmony of their symmetry, the geometry of their structure and the beauty of their colours. Exploring the rich properties of complex polynomials, we came across similar patterns that amazed us with the simplicity of their construction and the complexity of their structure. The idea is simple: transform a simple shape such as a circle or a rectangle in the plane by applying a complex polynomial whose set of zeros shares an axis of symmetry with the shape. The resulting *complex mandalas* exhibit a symmetry pattern closely related to the symmetry of the polynomial's zeros. In fact the underlying explanation is nothing else but an application of the Schwarz reflection principle stating that under some assumptions a holomorphic function can be reflected across a line. We will show how this classical result explains our complex mandalas and use it to create shapes similar to traditional mandalas.

Figure 1: *Mandala in an Indian Hindu temple.*

Complex Polynomials and their Reflections

The protagonists of our study are complex polynomials of arbitrary degree $n \in \mathbb{N}$, i.e. functions $p : \mathbb{C} \to \mathbb{C}$ of the type $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with complex coefficients $a_j \in \mathbb{C}$. By the fundamental theorem of algebra every such polynomial factors as a product $p(z) = a_n (z - \zeta_1)^{k_1} \cdot \ldots \cdot (z - \zeta_m)^{k_m}$ with $\sum_j k_j = n$. The numbers $\zeta_j \in \mathbb{C}$ are the *roots* or *zeros* of p and the exponents k_j their corresponding *multiplicities*. As an immediate consequence every complex polynomial is, up to a scalar, uniquely determined by its zeros and their multiplicities. We denote by $\mathbb{H}^+ := \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ and $\mathbb{H}^- := \{z \in \mathbb{C} : \mathrm{im}(z) < 0\}$ the upper and lower half plane, respectively and by $R := \mathbb{C} \cap \mathbb{R}$ and $I := \mathbb{C} \cap i\mathbb{R}$ the real and imaginary line embedded in $\mathbb{C}$. Central to all what follows is the *Schwarz reflection principle* (for details see, e.g., [1] or [2]) :

Theorem. *Let $U \subseteq \mathbb{C}$ be an open, connected set closed under complex conjugation, i.e. $z \in U \Rightarrow \bar{z} \in U$, let $U^+ := U \cap \mathbb{H}^+$, $U^0 := U \cap R$, $U^- := U \cap \mathbb{H}^-$. Then every continuous function $f : U^+ \cup U^0 \to \mathbb{C}$ with $f(U^0) \subseteq R$ and holomorphic, i.e. complex differentiable, on U^+ can be extended uniquely to a holomorphic function $F : U \to \mathbb{C}$ by $F(z) := f(z)$ for $z \in U^+ \cup U^0$, and $F(z) = \overline{f(\overline{z})}$ for $z \in U^-$.*

For a given line $L \subset \mathbb{C}$ we call a set $U \subseteq \mathbb{C}$ *L-symmetric*, if L is an axis of symmetry for U, i.e. if for every point $z \in U$ its reflection across L is also in U. Since affine transformations of the type $\varphi(z) = az + b$,

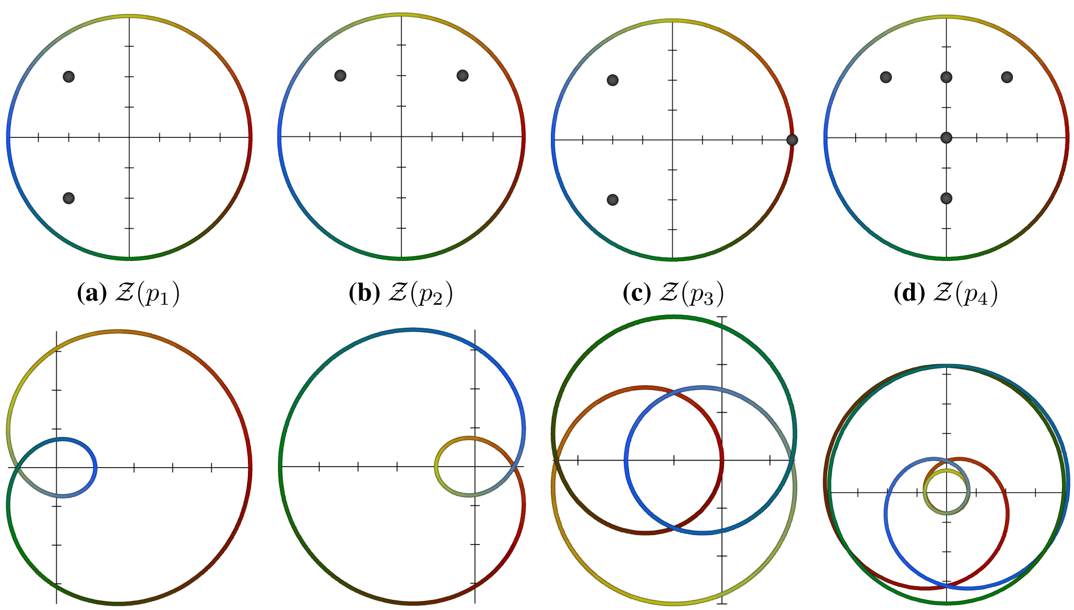
$a, b \in \mathbb{C}$, $a \neq 0$, map L-symmetric sets to $\varphi(L)$-symmetric sets the reflection principle can be stated slightly more general as

Corollary. *Let $L \subset \mathbb{C}$ be a line, $U \subseteq \mathbb{C}$ be an open, connected L-symmetric set, $U^0 := U \cap L$, and let U^+ and U^- denote the subsets of U to the left and right hand side of L, respectively (pick an arbitrary orientation of L). If $f : U^+ \cup U^0 \to \mathbb{C}$ is a continuous function with $f(U^0)$ contained in a line $\hat{L}$ and holomorphic on U^+, then f has a unique holomorphic extension $F : U \to \mathbb{C}$, and $F(U)$ is $\hat{L}$-symmetric.*

If p is a complex polynomial let $\mathcal{Z}(p)$ denote the (finite) set of its zeros. Assume there is a line $L \subset \mathbb{C}$ such that $\mathcal{Z}(p)$ is L-symmetric and every zero has the same multiplicity as its reflected counterpart. Then every L-symmetric set U is mapped to an $\hat{L}$-symmetric set $\hat{U} := p(U)$, where $\hat{L}$ is a line through $p(L)$. Indeed there is an affine biholomorphic transformation ψ that maps R to L and R-symmetric sets to L-symmetric sets. Therefore $\hat{p} := p \circ \psi$ is a polynomial whose set of zeros $\mathcal{Z}(\hat{p}) = \psi^{-1}(\mathcal{Z}(p))$ is R-symmetric. Then $\hat{p}$ is of the type $\hat{p}(z) = c \prod_j (z - t_j) \prod_k (z - b_k)(z - \overline{b_k}) = c \prod_j (z - t_j) \prod_k (z^2 - 2\mathrm{re}(b_k) z + \|b_k\|^2)$, where $t_j \in \mathcal{Z}(\hat{p}) \cap R$, $b_k, \overline{b_k} \in \mathcal{Z}(\hat{p}) \setminus R$ and $c \in \mathbb{C}$. If $c \in R$ then $\hat{p}(R) \subset R$, otherwise $\hat{p}(R)$ is contained in the line $\hat{L}$ passing through the origin and forming an angle $\arg c$ to R, so $p(L) = \hat{p} \circ \psi^{-1}(L) \subseteq \hat{L}$ and we can apply the corollary.

Symmetries, Construction and Colouring of Complex Mandalas

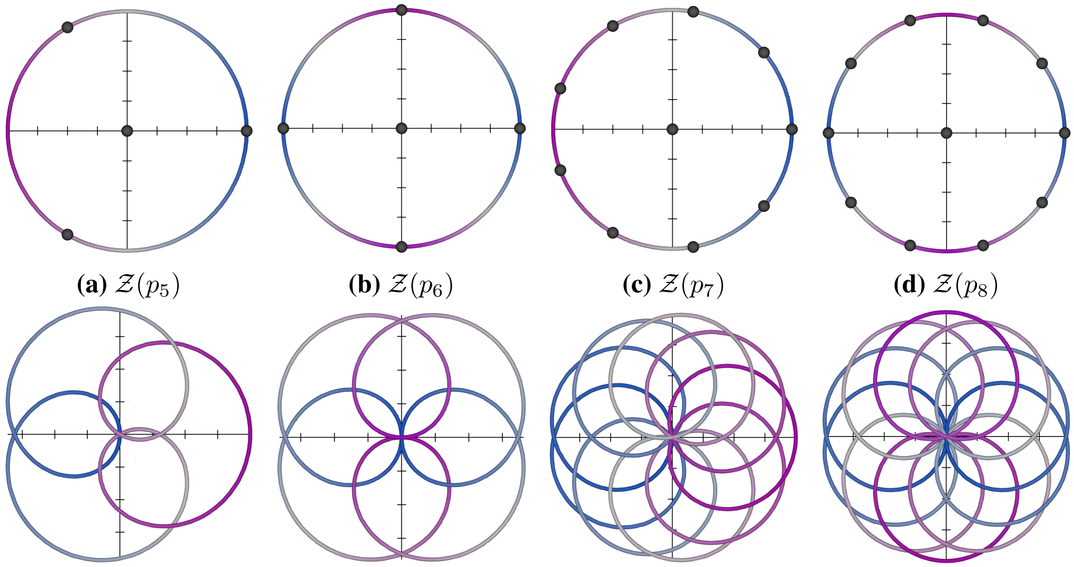
In the following we will focus on concentric circles C_r and regular N-gons R_r^N of radius r as L-symmetric sets, so that each symmetry axis L of the zero set $\mathcal{Z}(p)$ of a polynomial p must pass through the origin. Figures 2 and 3 demonstrate such examples that explain how the symmetry axis through $p(L)$ depends on the zeros of p, and how we acquire the mandala-resembling patterns.



(a) $\mathcal{Z}(p_1)$ **(b)** $\mathcal{Z}(p_2)$ **(c)** $\mathcal{Z}(p_3)$ **(d)** $\mathcal{Z}(p_4)$

(e) $p_1(C_1)$ **(f)** $p_2(C_1)$ **(g)** $p_3(C_1)$ **(h)** $p_4(C_1)$

Figure 2: *The image shows the zero sets $\mathcal{Z}(p_i)$ (top row) and the transformed unit circle C_1 for the polynomials $p_1(z) = 0.5 + z + z^2$, $p_2(z) = -0.5 - iz + z^2$, $p_3(z) = -0.5 - 0.5z + z^3$ and $p_4(z) = z^5 - iz^4 - 0.25z^3 - 0.25iz^2 - 0.125z$. Observe that both $p_1(C_1)$ and $p_2(C_1)$ are R-symmetric even though $\mathcal{Z}(p_2)$ is only I-symmetric. More generally if $\mathcal{Z}(p)$ is I-symmetric then $p(C)$ is R-symmetric if the degree of p is even, and I-symmetric otherwise, see $p_4(C_1)$.*

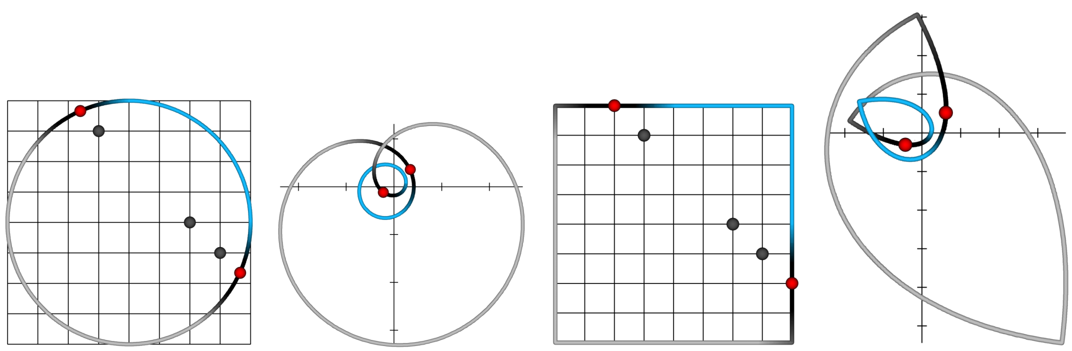


(a) $\mathcal{Z}(p_5)$ **(b)** $\mathcal{Z}(p_6)$ **(c)** $\mathcal{Z}(p_7)$ **(d)** $\mathcal{Z}(p_8)$

(e) $p_5(C_1)$ **(f)** $p_6(C_1)$ **(g)** $p_7(C_1)$ **(h)** $p_8(C_1)$

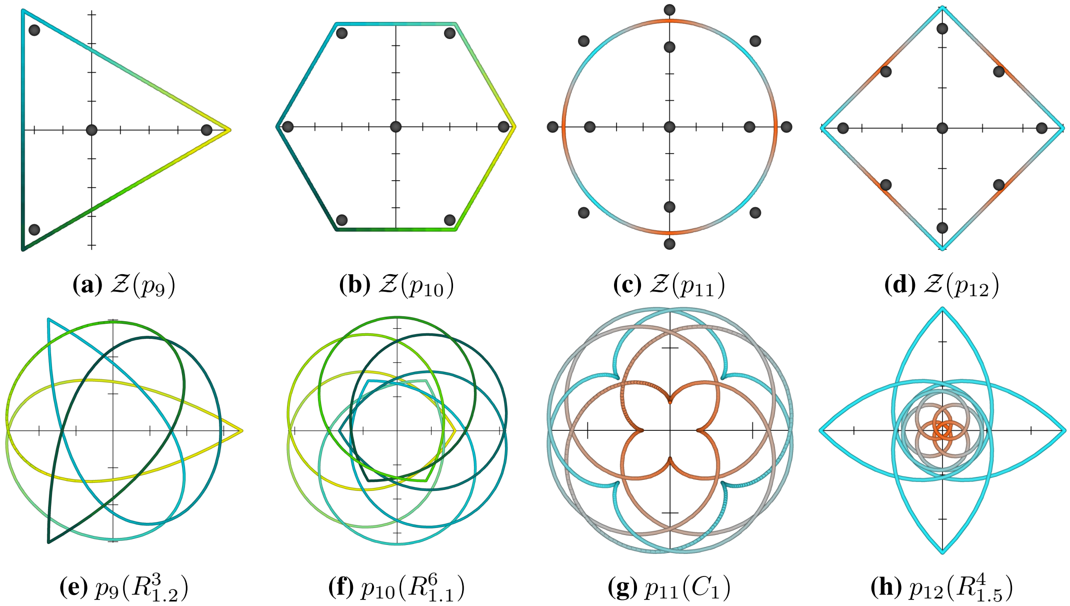
Figure 3: *The polynomials* $p_5(z) = z(z^3 - 1)$, $p_6(z) = z(z^4 - 1)$, $p_7(z) = z(z^9 - 1)$ *and* $p_8(z) = z(z^{10} - 1)$. *The roots of polynomials of the form* $z(z^n - 1)$ *are* $\zeta_0 = 0$ *and the* n*-th roots of unity* $\zeta_k = \exp(2\pi i k/n)$, $k = 1, \ldots, n$, *and every line through* ζ_0 *and a point* ζ_k *is an axis of symmetry for* $\mathcal{Z}(p_j)$. *Note that symmetry with respect to* R *and* I *is reflected by the chosen colour scheme. We refer the reader to the coloured version on the conference CD.*

Figure 4 shows that asymmetric zeros of p may result in asymmetries of the transformed curve. Further on in Figure 5 we present some more elaborated examples and the ideas behind their construction. Ultimately, since vivid colours are an important factor in the mandala art, we use winding numbers of the transformed curve to assign colours to the regions bounded by self-intersections of the curve, and demonstrate some of our math-made mandalas in Figure 6.



(a) $\mathcal{Z}(p)$ **(b)** $p(C_1)$ **(c)** $\mathcal{Z}(p)$ **(d)** $p(Q)$

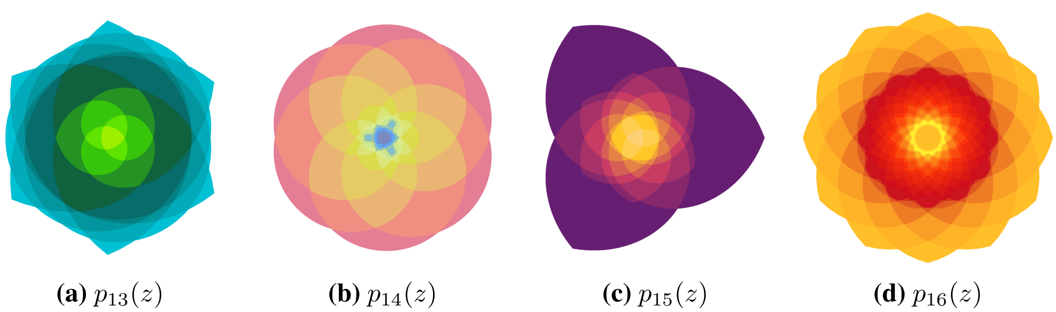
Figure 4: *The normalized polynomial* p *with zeros* $0.75 - 0.25i$, 0.5 *and* $-0.25 + 0.75i$. *Note that* C_1 *is not symmetric with respect to the line* L *through* $\mathcal{Z}(p)$, *resulting in an asymmetric transformation* $p(C_1)$ *(left two pictures). This is even more clear when one considers the square* Q *with side length* 2 *centered at* 0 *(right two pictures). However the red points* $C_1 \cap L$ *and* $Q \cap L$ *are still mapped to points on a line through the origin.*



(a) $\mathcal{Z}(p_9)$ **(b)** $\mathcal{Z}(p_{10})$ **(c)** $\mathcal{Z}(p_{11})$ **(d)** $\mathcal{Z}(p_{12})$

(e) $p_9(R^3_{1.2})$ **(f)** $p_{10}(R^6_{1.1})$ **(g)** $p_{11}(C_1)$ **(h)** $p_{12}(R^4_{1.5})$

Figure 5: *The four pictures on the left show the polynomials* $p_9(z) = z(z^3-1)$, $p_{10}(z) = z(z^3-1)(z^3+1) = z^7 - z$, *that have prescribed zeros at the 3rd and 6th roots of unity and the origin, and how they transform the corresponding polygons* $R^3_{1.2}$ *and* $R^6_{1.1}$. *The remaining four pictures on the right show the polynomials* $p_{11}(z) = \int (z^4 - \frac{1}{16})(z^8-1) = \frac{1}{13}z^{13} - \frac{1}{144}z^9 - \frac{1}{5}z^5 + \frac{1}{16}z$ *and* $p_{12}(z) = \int (z^4-1)(z^4+\frac{1}{4}) = \frac{1}{9}z^9 - \frac{3}{20}z^5 - \frac{1}{4}z$. *Note that this time these polynomials are given as anti-derivatives (with vanishing constant term) of polynomials* p'_{11}, p'_{12} *with prescribed zeros. It is these zeros that cause* $p_{11}(C_1)$ *to form spikes in the blue and orange region, related to so-called* ramification *that occurs at these points.* p_{12} *transforms the regular* 4*-gon* $R^4_{1.5}$.



(a) $p_{13}(z)$ **(b)** $p_{14}(z)$ **(c)** $p_{15}(z)$ **(d)** $p_{16}(z)$

Figure 6: *The* winding number *of a curve with respect to a point is, simply stated, the number of times the curve winds around the point. For the above pictures we assigned colours to the regions between the transformed curve's self-intersections according to the winding numbers of the points in each region. The shown polynomials are* $p_{13}(z) = -iz - z^4 + iz^7 + z^{10}$, $p_{14}(z) = z + \frac{1}{4}z^4 - \frac{2}{7}z^7 - \frac{1}{5}z^{10} + \frac{1}{13}z^{13} + \frac{1}{16}z^{16}$, $p_{15}(z) = 8z + 3z^4 + 4z^7 + 5z^{10}$ *and* $p_{16}(z) = z(z^{24}+1)$, *using either circular or polygonal domains.*

References

[1] Z. Nehari. *Conformal Mapping*. Dover Books on Mathematics. Dover Publications, 1975.

[2] H.A. Schwarz. Ueber einige Abbildungsaufgaben. *Journal für die reine und angewandte Mathematik*, 70:105–120, 1869.