Regular map smoothing

Faniry Razafindrazaka, Konrad Polthier

Freie Universität Berlin
{faniry.razafi,konrad.polthier}@fu-berlin.de

Abstract A regular map is a family of equivalent polygons, glued together to form a closed surface without boundaries which is vertex, edge and face transitive. The commonly known regular maps are derived from the Platonic solids and some tessellations of the torus. There are also regular maps of genus greater than 1 which are traditionally viewed as finitely generated groups. RMS (Regular Map Smoothing) is a tool for visualizing a geometrical realization of such a group either as a cut-out in the hyperbolic space or as a compact surface in 3−space. It provides also a tool to make the resulting regular map more appealing than before. RMS achieves that by the use of a coloring scheme based on coset enumeration, a Catmull-Clark smoothing scheme and a force-directed algorithm with topology preservation.

Keywords: hyperbolic geometry; computational group theory; surface topology; computational geometry; object modelling.

1 Introduction

RMS is a software for visualizing and smoothing regular maps. Regular maps are on the one hand finitely generated discrete groups and on the other hand, they are genus g compact surfaces. RMS was initially developed to improve the visualization of existing regular maps but it also explores discrete groups in a very simple and visually pleasing way. RMS, a result of Faniry Razafindrazaka's master thesis [8], is written in Java and uses the JavaView library (www.javaview.de) for visualization. In Figure 1 are examples of regular maps generated by RMS.
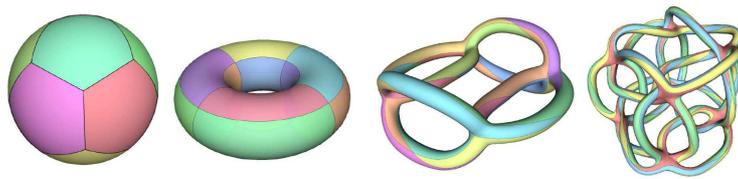

Figure 1: From left to right: regular maps of genus 0 (dodecahedron), 1 (torus $\{4,4\}_{4,0}$), 5 (R5.4′$\{6,4\}$) and 17 (R17.10′$\{12,4\}$) generated by RMS (image from [8]).

RMS gives access to a 3D representation of a discrete group as a compact orientable surface. As a summary, RMS gives access to the following visualization tools:

- Visualization of genus 0,1 regular maps (platonic solids, tori) with their color symmetry.

- Visualization of genus $g \geq 2$ regular maps as a finite discrete group G in the hyperbolic space.

- Enumeration of the coset of a subgroup H of G which induces a perfect coloring of G.

- Extension of the color symmetry of G to the universal covering of G which induces a perfect coloring of a tilling $\{p,q\}$ of the hyperbolic space.

- Finding of a $4g-$gon in the universal covering of G using a cut and paste algorithm, which defines a topological genus g surface.

- Generation of a 3D embedding of G as a compact genus g surface.

- Interactive manipulation of the embedded surface by a force-directed algorithm.

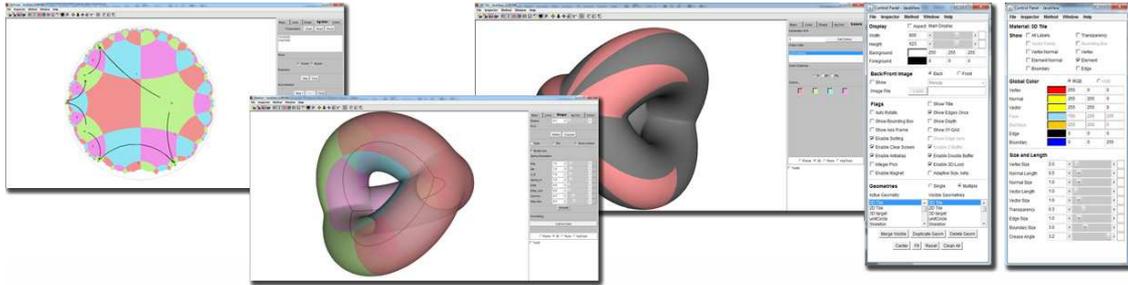An overview of the user interface of RMS is given in Figure 2.

Figure 2: Overview of the user interface proposed by RMS.

2   Related Works and Contributions

The visualization of regular maps is still a very challenging problem. The problem lies not only on finding 3D embeddings of the maps but also these embeddings should exhibit maximal symmetry. Up to now, only about 100 cases have been solved and there are still about 6000 left (referring to Conder list of regular maps [1]). Most of these solutions have been found by Jack van Wijk [5] with beautiful computer models. van Wijk's way of generating target surfaces is to take an existing regular map (for example: a platonic solid), turn its edges into tubes, its vertices into junctions, and its faces into holes. A *tubification* process, as he calls it, is a generalization of the tetrahedral tube frame of Carlos Séquin [11]. This process produces, in other words, regular surfaces and works well for the lower genus maps. Repeated application of this process may bring some degeneracies such as: self-intersected tubes or poor mesh quality (Figure 3, left) at the junctions. One way to solve these issues is to use small radii, but then the regular tilling mapped onto the surface becomes hard to visualize. These are cited as open problems in [5]. RMS was developed to solve these issues. As it is shown in Figure 3, RMS produces more appealing shapes than the direct tubification without further processing. For further investigation of the algorithm used in RMS, we refer to [5] and [8].
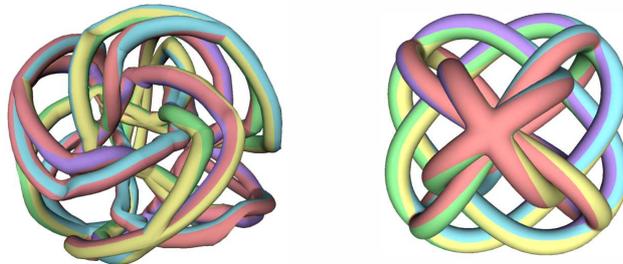

Figure 3: R13.2$'\{12, 3\}$ generated by a direct tubification of R3.4$'\{6, 4\}$ on the left and the smoothed version generated by RMS on the right.

3 Overview of RMS

The usage of RMS is intuitive and offers several tools for discrete group visualization, surface topology, color symmetry and spring energy minimization. It offers 2D and 3D viewing of a regular map which are perfectly in sync with respect to color symmetry and labelling. RMS takes a discrete group G of the form $\langle R, S, T | R^p, S^q, T^2, (RS)^2, (ST)^2, (RT)^2, g_1(R, S, T), \ldots, g_n(R, S, T) \rangle$ as input. It then performs a coset enumeration using a third party program developed by George Havas and Colin Ramsay [4], called ACE for Advance Coset Enumerator. Each element of a group G is represented in RMS first, as a combination of the letters `r`, `s` and `t`, and second, as a triangle. RMS has five panels described as follows;

-Maps offers the choice of a regular map to be visualized. It takes initially all the regular map which can be embedded, according to [5], from the Conder's list of maps [1]. The regular maps can be sorted by their genus starting from 0 to 101. It also includes a choice of target surfaces for regular maps which have several embeddings.

-Lines gives access to all edges of the regular maps including those of the dual regular maps. It also gives the user the choice of labelling of each triangle according to their representation in the coset table or by a simple numbering of the element representatives.

-Shapes is only available in the 3D viewing. It allows the user to modify the shape of the 3D embedding including: the radius, the number of discretization (refining/coarsening), extraction of the medial axis of the tube, a force-directed modelling and a Catmull-Clark smoothing scheme. The refining and coarsening are done using the DLG refinement scheme [3], the force-directed algorithm is the one presented in [9] and the Catmull-Clark smoothing scheme is the one from [7] which applies to the tube. The tile and the tube can be switched off during the smoothing process for fast rendering. Some vertices can be pulled manually by the user to avoid the system to be stuck in local minima.

-Colors is the panel for assigning a color symmetry to the regular map. The technique applied here is the one suggested by [10] and [2]. The default coloring scheme assigns a unique color to each coset of a subgroup $H = \langle r, t \rangle$ of G. Other coloring schemes, such as the minimal number of colors needed to perfectly color a hyperbolic tiling is also supported.

-4g-gon is a panel for 2D surface topology. For a given regular map in the hyperbolic disc, it starts by checking which side of the map has to be identified to give the compact surface. Then, it applies a cut and paste algorithm, described in [12], until a normalized 4g-gon is obtained. This 4g-gon can then be exported and manipulated for further modelling.

References

[1] Conder M. and Dobcsanyi P.: Determination of all Regular Maps of Small Genus, J. Combinatorial Theory, Series B 81 (2001), 224-242.

[2] de las Penas M.L.A.N., Felix R.P., Laigo G.R: Colorings of Hyperbolic Plane Crystallographic Patterns, (2006), Zeitschrift für Kristallographie, 221:665-672.

[3] Dyn N., Levin D. and Gregory J. A.: A 4-point Interpolatory Subdivision Scheme for Curve Design. Computer Aided Geometric Design 4 (1987) 257-268.
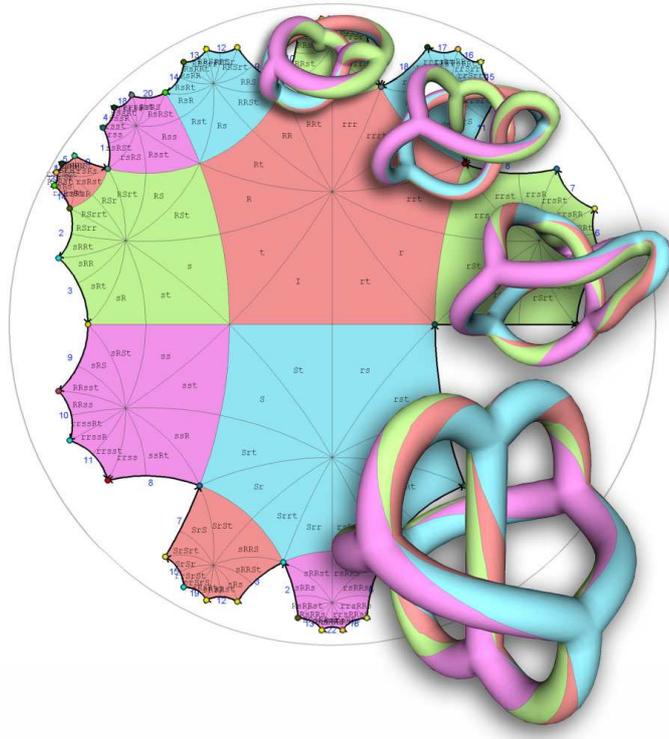
Figure 4: R4.3'{6, 4}, starting from the hyperbolic disc through a first candidate 3D embedding from [5] and smoothed by RMS to give an octahedral looking shape.

[4] Havas G. and Ramsay C.: ACE-Advanced Coset Enumerator, (2002), http://www.itee.uq.edu.au/ cram/ce.html

[5] van Wijk J. J.: Symmetric Tiling of Closed Surfaces: Visualization of Regular Maps. Proc. SIGGRAPH 2009 Conf., New Orleans, pp 49:1-12.

[6] van Wijk J. J. and Cohen A. M.: Visualization of Seifert Surfaces, (2006), IEEE Transactions on Visualization and Computer Graphics 12, 4,485-496.

[7] van Wijk J. J. and Cohen A. M.: Visualization of the Genus of Knots, (2005), Proc. IEEE Conf. Visualization, pp. 567-574.

[8] Razafindrazaka F.: Visualization of High Genus Regular Maps, master's thesis, Freie Universität Berlin, 2012, http://page.mi.fu-berlin.de/faniry/files/faniry_ masterThesis_ 2012.pdf.

[9] Scharein R. G.: Interactive Topological Drawing, (1998), Phd Thesis, The University of British Coloumbia.

[10] Senechal M.: Color Symmetry, (1988), Comput. Math. Applic., 16:545-553.

[11] Séquin C. H.: Patterns on the Genus-3 Klein quartic, (2006), In Proc. BRIDGES, London, pages 245-254.

[12] William S. Massey: A Basic Course in Algebraic Topology, (1991), Graduate Texts in Mathematics, Springer.