(Generalized) Wintgen Inequality

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In 1979, P. Wintgen [6] proved that the Gauss curvature K, the squared mean curvature $\|H\|^2$ and the normal curvature $K^\perp$ of any surface M^2 in $\mathbf{E}^4$ always satisfy the inequality

$$K \leq \|H\|^2 - |K^\perp|$$

and that actually the equality holds if and only if the ellipse of curvature of M^2 in $\mathbf{E}^4$ is a circle.

In 1999, P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken [1] formulated the conjecture on generalized Wintgen inequality which is also known as the *DDVV conjecture*.

Conjecture. *Let $f : M^n \to \widetilde{M}^{n+m}(c)$ be an isometric immersion, where $\widetilde{M}^{n+m}(c)$ is a real space form of constant sectional curvature c. Then*

$$\rho \leq \|H\|^2 - \rho^\perp + c,$$

where ρ is the normalised scalar curvature (intrinsic invariant) and $\rho^\perp$ is the normalised normal scalar curvature (extrinsic invariant).

Recently, the DDVV-conjecture was finally settled for submanifolds in real space forms, of arbitrary dimension and codimension, by Z. Lu [3] and independently by J. Ge and Z. Tang [2].

Recently we obtained DDVV inequalities, also known as generalized Wintgen inequalities, for Lagrangian submanifolds in complex space forms [4] and Legendrian submanifolds in Sasakian space forms [5], respectively. Some applications are given. Also we stated such inequalities for slant submanifolds in complex space forms and Sasakian space forms, respectively. Further developments are mentioned.

References.

[1] P.J. De Smet, F. Dillen, L. Verstraelen, L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. (Brno) **35** (1999), 115–128.

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[3] Z. Lu, *Normal scalar curvature conjecture and its applications*, J. Funct. Analysis **261** (2011), 1284-1308.

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